C, P, T conjugation formulas for spinor field operators

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CPT symmetry

CPT symmetry plays a key role in modern quantum field theory (QFT). It guarantees for particles and antiparticles equality of masses, decay widths, equality in absolute value and opposite sign of all types of charges.

CPT:

- is **anti-unitary** operator,
- preserves the Dirac equation $\gamma^{\mu}(i\partial_{\mu} - qA_{\mu}(x))\,\psi(x) = m\psi(x)$,
- particle → antiparticle, charge $q \to -q$, potential $A_{\mu} \to -A_{\mu}$,
- energy $p_0 \to p_0$,
- momentum $p_k \to p_k$ (i.e. $\boldsymbol{p} \to \boldsymbol{p}$),
- spin projection $s_3 \to -s_3$,
- **chirality L→R, R →L**, where $\psi_L = \frac{1-\gamma^5}{2}\psi,\quad \psi_R = \frac{1+\gamma^5}{2}\psi$

(L - left-chiral, R - right-chiral. Chirality and helicity should not be confused!)

General properties of C, P, T conjugations (inversions)

Operators **C, P, T**

- preserve the Dirac equation $\gamma^{\mu}(i\partial_{\mu} - qA_{\mu})\, \psi = m\psi$,
- preserve energy $E = p_0 \rightarrow p_0$.

C: **unitary** operator, which transforms

- particle $\rightarrow$ antiparticle, charge $q \rightarrow -q$; potential $A_{\mu} \rightarrow -A_{\mu}$,
- momentum $\boldsymbol{p} \rightarrow \boldsymbol{p}$,
- spin projection $s_3 \rightarrow s_3$,
- **preserves chirality** L→L, R →R

P: **unitary** operator, which transforms

- particle → particle; charge $q \rightarrow q$; potential $A_0 \rightarrow A_0$, $\boldsymbol{A} \rightarrow -\boldsymbol{A}$,
- momentum $\boldsymbol{p} \rightarrow -\boldsymbol{p}$,
- spin projection $s_3 \rightarrow s_3$,
- **changes chirality** L→R, R →L

T: **anti-unitary** operator, which transforms

- particle → particle; charge $q \rightarrow q$; potential $A_0 \rightarrow A_0$, $\boldsymbol{A} \rightarrow -\boldsymbol{A}$,
- momentum $\boldsymbol{p} \rightarrow -\boldsymbol{p}$,
- spin projection $s_3 \rightarrow -s_3$,
- **preserves chirality** L→L, R →R

Erroneous formulas of CPT conjugation

Standard formulas of CPT conjugation are [1], [2], [5]

$$\mathrm{CPT}\ \psi(t,x) = \eta\ \gamma^5\ \psi(-t,-x), \quad (1)$$

or [2], [3], [4], [6], [7]

$$\mathrm{CPT}\ \psi(t,x) = \eta\ \gamma^5\ \psi^{+Tmatr}(-t,-x), \quad (2)$$

or [7], [8]

$$\mathrm{CPT}\ \psi(t,x) = \eta\ \gamma^5\ \psi^{*}(-t,-x) = \eta\ \gamma^5\ \psi^{+T}(-t,-x), \quad (3)$$

where $(.)^{Tmatr}$ is matrix transposition, and $(.)^{T} = (.)^{+*} = (.)^{+}\ (.)^{*}$

[1]. W. Pauli. Exclusion principle, Lorentz group and reflexion of space-time and charge. In: *Niels Bohr and the Development of Physics*. Pergamon Press, 30-51 (1955).

[2]. G. Lüders. Proof of the TCP Theorem. *Annals of Physics*, 2(1), 1-15 (1957).

[3]. G. Feinberg, S. Weinberg. On the Phase Factors in Inversions. *Il Nuovo Cimento*, 14, 571-592 (1959).

[4]. G.Grawert, G.Lüders, H. Rollnik. The TCP theorem and its applications. *Fortschritte der Physik*, *7*(6), 291-328 (1959).

[5]. J. Bjorken, S. Drell, Relativistic Quantum Mechanics, Mcgraw-Hill College: N.Y. (1964).

[6]. J. Bjorken, S. Drell, Relativistic Quantum Fields, Mcgraw-Hill College: N.Y. (1965).

[7]. M. Peskin, D. Schroeder. An Introduction to Quantum Field Theory, Addison-Wesley Publishing Company: USA (1995).

[8]. S. Weinberg. The Quantum Theory of Fields - Foundations. Vol. 1. Cambridge University Press: Cambridge, 1995.

Why these formulas are erroneous?

Let according to (1)

$\text{CPT}\ \psi(t,x) = \eta\ \gamma^5\ \psi(-t,-x)$,

=> $\text{CPT} = \eta\ \gamma^5\ R\ (-t,-x)$, where $R\ (-t,-x)\ \psi(t,x) = \psi(-t,-x)$.

R (-t,-x) = R (-t) R (-x) acts only on the t and x components, not on the matrices or creation and annihilation operators - it is very important!

=> $\text{CPT}\ \gamma^5 = \gamma^5\ \text{CPT}$

=> $\text{CPT}\ \psi_L(t,x) = \eta\ \gamma^5\ \psi_L(-t,-x) = (\eta\ \gamma^5\ \psi\,(-t,-x))_L$

This CPT operator transforms left-chiral spinor to left-chiral antispinor!

The same is for formulas (2) and (3) of the CPT.

According to the Standard Model there should be:

$\text{CPT}\ \psi_L = \psi^a{}_R$, where ψ^a – field operator of the antiparticle.

=> **formulas (1), (2), (3) are erroneous!**

Where is the error in the formulas for C, P, T?

1. Space inversion operator is unitary and changes chirality L→R, R →L:

$$\mathbf{P} = \boldsymbol{\eta}_{\mathbf{P}}\, \gamma^0\, \boldsymbol{R}\, (-\boldsymbol{x}) \qquad (4)$$

2. Time inversion operator is anti-unitary and preserves chirality L→ L, R → R

$$\mathbf{T} = \boldsymbol{\eta}_{\mathbf{T}}\, \gamma^1 \gamma^3\, \boldsymbol{R}\, (-\boldsymbol{t})\, (.)^* \qquad (5)$$

3. There are **four charge conjugation operators** (we use Dirac or Weil representation of gamma matrices): **non-QFT** operators $C_{M\text{-}K}$ and C_{Pauli}; **QFT** operators C_{Schw} and C_{QFT}.

- $C_{M\text{-}K} = \eta_C\, \gamma^2 (.)^*$ –Majorana-Kramers; **changes chirality, anti-unitary** => **error.**
- C_{Pauli}: $C_{M\text{-}K}$ and «hole» →antiparticle – **changes chirality, anti-unitary** => **error.**
- $C_{Schw} = \eta_C\, \gamma^2 (.)^{+\,Tmatr}$ – Schwinger charge conjugation is QFT analog of C_{Pauli}; **changes chirality, anti-unitary** => **error**;
- C_{QFT}: $b_s(p) \rightarrow \eta_C\, d_s(p)$, $d_s^{+}(p) \rightarrow \eta_C\, b_s^{+}(p)$; **preserves chirality, unitary**

=> C_{QFT} is the operator C of the CPT theorem!

The origin of the error: $C_{QFT}\, \psi = C_{Schw}\, \psi$ for non-chiral spinors

=> erroneous conclusion that "$C_{QFT} \equiv C_{Schw}$". However, $\mathbf{C_{QFT}}\, \boldsymbol{\psi_L} \neq \mathbf{C_{Schw}}\, \boldsymbol{\psi_L}$.

A well-known implementation of C_{QFT}

J. Bjorken, S. Drell [6] :

$$C_{QFT}\psi(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\sqrt{\frac{m}{E}}\left(Cb_s(p)C^{-1}u_s(p)e^{-ip_\mu x^\mu}+Cd_s^+(p)C^{-1}v_s(p)e^{ip_\mu x^\mu}\right)$$

$$C=e^{iA_1\varphi}e^{-iA_2\varphi}=e^{-iA_2\varphi}e^{iA_1\varphi};\varphi=\frac{\pi}{2}$$

$$A_1=A_1^+=\int d^3p\ \left(b_s^+(p)b_s(p)+d_s^+(p)d_s(p)\right);$$

$$A_2=A_2^+=\int d^3p\ \left(b_s^+(p)d_s(p)+d_s^+(p)b_s(p)\right);$$

$$C_{QFT}=\int d^3p\ [b_s^+(p)b_s(p)+d_s^+(p)d_s(p),\bullet]\int d^3p\ [b_s^+(p)d_s(p)+d_s^+(p)b_s(p),\bullet]$$

$$C_{QFT}b_s(p)=Cb_s(p)C^{-1}=d_s(p);\qquad C_{QFT}b_s^+(p)=d_s^+(p);$$

$$C_{QFT}d_s^+(p)=Cd_s^+(p)C^{-1}=b_s^+(p);\qquad C_{QFT}d_s(p)=b_s(p);$$

Errors in the formulas for P and T

- Is operator $\mathbf{C_{QFT}PT} = C_{QFT}\, \eta_P\, \eta_T\, R\,(-t,-x)\, i\, \gamma^2\gamma^5\, (.)^*$? It is anty-unitary, changes chirality L→R, R →L => is all correct? – **No**. Why?
- Previous authors (J. Bjorken, S. Drell; M. Peskin, D. Schroeder; S. Weinberg and so on) have suggested that the transformation of the spinor field operator can be described by
  - either matrix operators $S(p)$,
  - or transformations of operators $b_s(p)$, $d_s^+(p)$,

  and that these methods are equivalent.
- Let us prove that this assumption is wrong.

Proper Lorentz transformations

To obtain $\psi_s(p')$ from $\psi_s(p)$, it is necessary to apply active Lorentz transformation.

$$\psi_s(p) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{E(p)}}\left(b_{<s>}(p)\ u_{<s>}(p)\,e^{-ip_\mu x^\mu} + d^+_{<s>}(p)\,v_{<s>}(p)\,e^{ip_\mu x^\mu}\right). \qquad (6)$$

$$\psi_s(p') = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{E(p')}}\left(b_{<s>}(p')\,u_{<s>}(p')\,e^{-ip'_\mu x^\mu} + d^+_{<s>}(p')\,v_{<s>}(p')e^{ip'_\mu x^\mu}\right). \qquad (7)$$

=>

$$b_s(p) \to b_s(p'); \quad u_s(p) \to u_s(p'); \quad e^{-ip_\mu x^\mu} \to e^{-ip'_\mu x^\mu}, \; i\partial_\mu\, e^{-ip'_\mu x^\mu} = p'_\mu,$$

$$d^+_s(p) \to d^+_s(p'); \; v_s(p) \to v_s(p'); \quad e^{ip_\mu x^\mu} \to e^{ip'_\mu x^\mu}, \quad i\partial_\mu\, e^{ip'_\mu x^\mu} = -p'_\mu.$$

After the Lorentz transformation $p_\mu x^\mu \to p'_\mu x'^\mu$. However, we observe

the field operator (9) at the same point $x' = x$ and not in the point $x' = \Lambda x$.

That is why $p'_\mu x'^\mu = p'_\mu x^\mu$, and the field operator $\psi(x)$ transforms

into $\psi'(x)$, and not into $\psi'(\Lambda x)$.

Error in the formula for P

$\psi(x)$ **symmetrically contains terms with *p* and *–p, s* and *–s*** . Therefore, it is necessary to investigate transformations of $\boldsymbol{\psi_s(p)}$, and **not** $\boldsymbol{\psi(x)}$.

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3 p \sqrt{\frac{m}{E}} \left(b_s(p) u_s(p) e^{-ip_\mu x^\mu} + d_s^+(p) v_s(p) e^{ip_\mu x^\mu} \right) = \int d^3 p \sum_s \psi_s(p), \qquad (8)$$

$$\mathrm{P}\psi_s(p) = \frac{\eta_P}{(2\pi)^{3/2}} \sqrt{\frac{m}{E(p)}} \left(b_{<s>}(p) u_{<s>}(-p) e^{-ip_\mu \tilde{x}^\mu} - d_{<s>}^+(p) v_{<s>}(-p) e^{ip_\mu \tilde{x}^\mu} \right) =$$

$$= \frac{\eta_P}{(2\pi)^{3/2}} \sqrt{\frac{m}{E(\tilde{p})}} \left(b_{<s>}(p) u_{<s>}(\tilde{p}) e^{-i\tilde{p}_\mu x^\mu} - d_{<s>}^+(p) v_{<s>}(\tilde{p}) e^{i\tilde{p}_\mu x^\mu} \right) \neq \eta_P \psi_s(-p), \qquad (9)$$

where $\tilde{p}^0 = p^0;\ \tilde{p}^k = -p^k,\ k = 1, 2, 3.$

=> $\mathrm{P}\psi_s(p)$ is **invalid field operator** with $b_s(p)$ and $-d_s^+(p)$ instead of $b_s(-p)$ and $-d_s^+(-p)$

=> P must transform $b_s(p)$ to $b_s(-p)$ and $d_s^+(p)$ to $-d_s^+(-p)$!

Space inversion operator P

Thus, the action of the spatial inversion operator on the spinor field operator leads to

$$\psi'_s(p) = \frac{\eta_P}{(2\pi)^{3/2}}\sqrt{\frac{m}{E(\tilde{p})}}\left(b_{<s>}(\tilde{p})u_{<s>}(\tilde{p})e^{-i\tilde{p}_\mu x^\mu} + d^+_{<s>}(\tilde{p})v_{<s>}(\tilde{p})e^{i\tilde{p}_\mu x^\mu}\right),$$

$$\psi'(x) = \eta_P\int d^3\tilde{p}\sum_s \psi_s(\tilde{p}) = \eta_P\int d^3p\sum_s \psi_s(p) =$$

$$= \frac{\eta_P}{(2\pi)^{3/2}}\int d^3p\sqrt{\frac{m}{E}}\left(b_s(p)u_s(p)e^{-ip_\mu x^\mu} + d^+_s(p)v_s(p)e^{ip_\mu x^\mu}\right) = \eta_P\,\psi(x).$$

=>

$$\mathrm{P}\psi_s(p) = \eta_P\psi_s(-p),$$

$$\mathrm{P}\,\psi(x) = \eta_P\,\psi(x),$$

$$\mathrm{P}\,\psi_L(x) = \eta_P\,\psi_R(x),$$

$$\mathrm{P} = \eta_P\gamma^0 R(\Lambda),$$

where $\eta_P = \pm 1$ and $R(\Lambda)b_s(p) = b_s(-p)$, $\quad R(\Lambda)d^+_s(p) = -d^+_s(-p)$.

Formulas for proper Lorentz transformations and space-time inversions are similar

- $\mathbf{L} = \boldsymbol{\eta}_{\mathrm{L}} \boldsymbol{S}\, \boldsymbol{R}(\boldsymbol{\Lambda})$; $S = \eta_{\mathrm{LS}} \exp(\gamma^0 \gamma(p) \omega(p))$

  $R(\Lambda)\, b_s(p) = \eta_{\mathrm{Lb}} b_s(\Lambda^{-1} p)$; $R(\Lambda)\, d^+_s(p) = \eta_{\mathrm{Lb}} d^+_s(\Lambda^{-1} p)$; $\eta_{\mathrm{L}} = \eta_{\mathrm{Lb}} \eta_{\mathrm{LS}} = \eta_{\mathrm{Lb}} = \eta_{\mathrm{LS}} = 1$

- $\mathbf{P} = \boldsymbol{\eta}_{\mathrm{P}} \boldsymbol{S}\, \boldsymbol{R}(\boldsymbol{\Lambda})$; $S = \eta_{PS}\, \gamma^0$

  $R(\Lambda) b_s(p) = \eta_{\mathrm{Pb}} b_s(\Lambda^{-1} p) = \eta_{\mathrm{Pb}} b_s(-p)$; $R(\Lambda)\, d^+_s(p) = -\eta_{\mathrm{Pb}} d^+_s(-p)$; $\eta_{\mathrm{P}} = \eta_{\mathrm{Pb}}\, \eta_{PS} = \pm 1$.

  $\mathbf{P} \boldsymbol{\psi}_{\mathbf{s}}(p) = \boldsymbol{\eta}_{P} \boldsymbol{\psi}_{\mathbf{s}}(-p)$

  $\mathbf{P} \boldsymbol{\psi}(x) = \boldsymbol{\eta}_{P} \boldsymbol{\psi}(x)$; $\mathbf{P}^2 = 1$

- $\mathbf{T} = \boldsymbol{\eta}_{\mathrm{T}} \boldsymbol{S}\, (.)^* \boldsymbol{R}(\boldsymbol{\Lambda})$; $S = \eta_{TS} \gamma^1 \gamma^3$

  $R(\Lambda)\, b_s(p) = \eta_{\mathrm{Tb}} b_{-s}(-p)$; $R(\Lambda)\, d^+_s(p) = \eta_{\mathrm{Td}} d^+_s(-p)$; $\eta_T = \eta_{\mathrm{Tb}}\, \eta_{TS}$

  $\mathbf{T} \boldsymbol{\psi}_{\mathbf{s}}(p) = \boldsymbol{\eta}_{T} (-s)\, \boldsymbol{\psi}_{-\mathbf{s}}(-p)$; $\mathbf{T}^2 = -1$

CPT transformation

$C_{QFT}\,\psi \equiv \psi^a$ – antiparticle field operator.

- CPT $=\eta_{CPT}\, R(-t,-x)\, C_{QFT}\, \gamma^5\, i\, \gamma^2\, (.)^*$

- **CPT$\boldsymbol{\psi_s(p) = \eta_{CPT}(-s)\psi^a_{-s}(p)}$**
- **CPT$\boldsymbol{\psi_{sL}(p) = \eta_{CPT}(-s)\psi^a_{-sR}(p)}$**

Conclusions

- "Standard" formulas for the CPT transformation are invalid: they transform a left-chiral spinor into a left-chiral antispinor.
- Correct operator C of the charge conjugation is $\mathbf{C_{QFT}}$ such as
  $C_{QFT}\, b_s(p) = d_s(p)$; $\quad C_{QFT}\, d_s(p) = b_s(p)$.
- Lorentz transformations acts both on the spinor columns u_s, v_s via the matrix operator S, and on the creation and annihilation operators $b_s(p)$ and $d_s^{+}(p)$ via the transformation of their argument $p_\mu \rightarrow p'_\mu$ and possible phase factor.
- Base on this we derived the correct formulas for **P**, **T** and **CPT** operators which correspond to the Standard model.