Valent Quark Effective Model for Hadrons on the Light Front (LF)

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The problem considered here is the construction of effective Hamiltonian on the light front which can describe the hadron spectrum and structure.
At that we use the quark model with valence quarks.
The LF we describe with the following coordinates:

$$x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}}, \qquad x^{\perp} = (x^1, x^2), \tag{1}$$

where x^+ plays the role of time and the LF is determined by the condition $x^+ = 0$.

Quantization on the light front allows to introduce simple description of physical vacuum as the lowest state of kinematic momentum operator $P_{-} \geqslant 0$ (at $m^2 \geqslant 0$) and construct Fock space on the LF.

The other advantage of the quantization on the LF is the possibility to transform the state of particle in the rest frame to it's state with arbitrary momentum, at that staying on the LF (this Lorentz transformation corresponds to generators M_{+-}, $M_{\perp -}$).

When we constructing states, we introduce the basis of gauge invariant states, which include the fields of quarks on the LF, and important that their spatial coordinates are separated one from the other and then we use P-ordered exponent of gluon fields (A_μ) for the "color" group $SU_c(3)$:

$$U_{x_1,x_2} = P\exp\left(-i\int_{x_1}^{x_2} dx^\mu A_\mu(x)\right). \tag{2}$$

For meson states we have the following expression:

$$\{\psi_+^{+i}(x_1)(U_{x_1,x_2})^{ij}\psi_+^j(x_2)|0\rangle\}_{x_i^+=0},$$

where $\psi_+^i(x)$ is the independent component of bispinor $\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$.

And this is expression for baryon states:

$$\{\psi_+^{+i}(x_1)(U_{x_1,x_0})^{ii'}\psi_+^{+j}(x_2)(U_{x_2,x_0})^{jj'}\psi_+^{+k}(x_3)(U_{x_3,x_0})^{kk'}\varepsilon_{i'j'k'}|0\rangle\}_{x_i^+=0},$$

where the coordinates of fields are connected to an arbitrary point x_0; for example, one can choose $x_0 = \frac{1}{3}(x_1 + x_2 + x_3)$.
To simplify the model, we take the simplest form of gluon fields A_μ, homogenious in space coordinates and diagonal in color indices $A_\mu^{ij}(x) = \delta^{ij} A_{i\mu}(x^+)$. Then

$$(U_{x,x'})^{ij}_{x^+=x'^+=0} = \delta^{ij} e^{iA_{i\perp}(x-x')^\perp} e^{iA_{i-}(x-x')^-}, \tag{5}$$

where $A_{i\mu}(x) \equiv A_\mu^{ii}(x), \quad \sum_{i=1}^3 A_{i\mu} = 0.$

We start with the construction of hadron states in the rest frame and take the concrete wave function, namely, as the eigen function of the spectral problem which approximates the experimental spectrum of hadrons.
Consider, for example, meson state:

$$|\,\overset{o}{p}\,\rangle_{y^+=0} = \int d^{\,[3]}\bar{y}\, d^{\,[3]}y \sum_{i=1}^{3} \psi^{i+}(y_1)\psi^{i}(y_2)|0\rangle e^{iB^i y} e^{-i\overset{o}{p}\bar{y}} f(\mathbf{y}), \tag{6}$$

where $p^\mu = \overset{o}{p}{}^\mu$, $\overset{o}{p}{}^\mu = (m, \mathbf{0})$, m is the meson mass,
$\bar{y} = \frac{1}{2}(y_1 + y_2)$, $y = y_1 - y_2$, $d^{\,[3]}y = d^2 y^\perp dy^-$,
$B^i_\mu \equiv B^{ij}_\mu \delta_{ij}$ is the gluon field in hadron rest frame, and the wave function $f(\mathbf{y})$ is defined as the eigenfunction of equation for 3-dimensional quantum harmonic oscillator:
$(-\triangle_y + \beta^2 \mathbf{y}^2)\, f(\mathbf{y}) = \mu^2 f(\mathbf{y})$, $\mathbf{y} = (y^\perp, \frac{y^-}{\sqrt{2}})$, and μ^2 is the eigenvalue corresponding to the mass squared of meson.

The meson state with arbitrary momentum $p_\mu = (\Lambda(\alpha_p))_\mu{}^\nu \overset{o}{p}_\nu$ can be constructed using the following transformation:

$|p\rangle_{x^+=0} = U(\alpha_p) | \overset{o}{p} \rangle_{y^+=0}$, where

$U(\alpha_p) = e^{iM_{-k} \frac{p^\perp}{p_-}} e^{iM_{+-} \ln \frac{p_-}{m}}$, $\alpha_p = \begin{pmatrix} \sqrt{\frac{p_-\sqrt{2}}{m}} & 0 \\ \frac{p^{1+i2}}{\sqrt{mp_-\sqrt{2}}} & \sqrt{\frac{m}{p_-\sqrt{2}}} \end{pmatrix}$ is

the matrix of this transformation in fundamental representation. In this case, B^i_μ goes to A^i_μ, $B^i_\nu = \left((\Lambda(\alpha_p))^{-1} \right)_\nu{}^\mu A^i_\mu$:

$$B_{i\perp} = A_{i\perp} - \frac{P_\perp}{P_-} A_{i-}, \quad B_{i-} = \frac{m\, A_{i-}}{P_-\sqrt{2}}, \tag{7}$$

The corresponding transformation of coordinates has the following form: $y^\mu = \left(\Lambda^{-1}(\alpha_p)\right)^\mu{}_\nu \, x^\nu, \quad y^\perp = x^\perp,$
$y^- = \frac{\sqrt{2}}{m}\left(p_- x^- + p_\perp x^\perp\right).$
The quark field is transformed as follows:

$$U(\alpha_p)\,\psi(y)\,U^{-1}(\alpha_p) = \sqrt{\frac{\overset{o}{p}_-}{p_-}}\,\psi(x), \qquad \overset{o}{p}_- = \frac{m}{\sqrt{2}}.$$

And the integration measure is transformed as $d^{[3]}y = \frac{\sqrt{2}}{m}\, p_-\, d^{[3]}x$.
So we get: $|p\rangle_{x^+=0} = p_- \frac{\sqrt{2}}{m} \times$
$\times \int d^{\,[3]}\overline{x}\; d^{\,[3]}x\; \sum_{i=1}^{3} \psi^{i+}(x_1)\,\psi^{i}(x_2)\,|0\rangle \exp(iA^i x)\, \exp(-ip\overline{x})\, f(\mathbf{y}(x)).$

To simplify the problem of constructing effective Hamiltonian, we move from fields to creation operators:

$$|p,\lambda\rangle_{x^+=0} = \int d^{[3]}x\, d^{[3]}\bar{x}\, d^{[3]}k\, d^{[3]}\bar{k} \times$$

$$\times \sum_{i=1,2,3;\ s_1,s_2=\pm 1/2} b^{i+}_{s_1}(k_1)\, d^{i+}_{s_2}(k_2)\, |0\rangle\, \exp\left(iA_i x\right) \times$$

$$\times \exp\left\{ i\left((\bar{k} - p)\bar{x} + kx \right) \right\}\, f_{s_1,s_2,\lambda}(\mathbf{y})\,, \qquad (8)$$

where $\bar{x} = \frac{1}{2}(x_1 + x_2)$, $x = x_1 - x_2$, $\bar{k} = k_1 + k_2$, $k = \frac{1}{2}(k_1 - k_2)$, p is the momentum of hadron state, $d^{[3]}x \equiv dx^- d^{(2)}x^\perp$, $s_1,\, s_2,\, \lambda$ are indices associated with the spin of quarks, $b^{i+}_s(k) \equiv b^{i+}(k,s)$, $d^{i+}_s(k) \equiv d^{i+}(k,-s)$.

Carrying out the similar procedure for baryon states, we obtain:

$$|p,\lambda\rangle_{x^+=0} = \int d^{[3]}x\, d^{[3]}k\, d^{[3]}\overline{x}\, d^{[3]}\overline{k}\, d^{[3]}\widetilde{x}\, d^{[3]}\widetilde{k} \left(\prod_{i=1}^{3} b^{i+}_{s_i}(k_i)\right) |0\rangle \times$$

$$\times \exp\left(iA_i x_i\right) \exp\left\{i((\overline{k}-p)\overline{x} + kx + \widetilde{k}\widetilde{x})\right\} f_{s_1,s_2,s_3,\lambda}(\mathbf{y},\widetilde{\mathbf{y}}) \;, \qquad (9)$$

where $\overline{x} = \frac{1}{3}(x_1+x_2+x_3), \quad x = x_1 - x_2,$
$\widetilde{x} = \frac{1}{2}(x_1+x_2) - x_3, \quad y = y_1 - y_2, \quad \widetilde{y} = \frac{1}{2}(y_1+y_2) - y_3,$
$\overline{k} = k_1+k_2+k_3, \quad k = \frac{1}{2}(k_1-k_2), \quad \widetilde{k} = \frac{1}{3}(k_1+k_2-2k_3).$

As a result, it is possible to find the effective mass squared operator for which these states turned out to be eigenstates:

$$P^2 = \varkappa \sum_{i=1,2,3;\, s=\pm 1/2} \int d^3q \left\{ b_s^{i+}(q)\, b_s^i(q)\, (\mathbf{u} + \mathbf{B}_i)^2 + \right.$$

$$\left. + d_s^{i+}(q)\, d_s^i(q)\, (\mathbf{u} - \mathbf{B}_i)^2 \right\} - \tfrac{1}{3}\, g^2\, (\nu + 1)\, (\triangle_{\mathbf{B}} + \tfrac{4}{3}\, \triangle_{\widetilde{\mathbf{B}}}) +$$

$$+ \nu M_0^2 - \frac{\nu}{3}\, (\nu + 1)\, \sqrt{6\varkappa}\, g,$$

where

$$\mathbf{B} = \frac{1}{2}\, (\mathbf{B}_1 - \mathbf{B}_2)\,, \quad \widetilde{\mathbf{B}} = \mathbf{B}_1 + \mathbf{B}_2, \quad \sum_{i=1}^{3} \mathbf{B}_i = 0.$$

$$u_\perp(q) = \left(q_\perp - \frac{P_\perp}{P_-} q_- \right), \quad u_3(q) = m \left(\frac{1}{\nu} - \frac{q_-}{P_-} \right),$$

and ν and m mean the operators of total number of quarks and antiquarks and the mass, respectively.

We can check that our states are eigenstates for P^2. For example, for mesons:

$$P^2|p,\lambda\rangle_{x^+=0} = \int d^{[3]}x\, d^{[3]}k \left(\sum_{i=1,2,3;\ s_1,s_2=\pm\frac{1}{2}} b^{i+}_{s_1}(k_1)\, d^{i+}_{s_2}(k_2)\,|0\rangle \times \right.$$

$$\left. \times \exp\left(i(k+A_i)x\right) \right) \left(\frac{4}{3}\, g^2\, \mathbf{y}^2 - 2\varkappa \triangle_{\mathbf{y}} + 2M_0^2 \right) f_{s_1,s_2,\lambda}(\mathbf{y}).$$

Using the known spectrum of 3-dimensional quantum harmonic oscillator, we obtain the following spectrum of mass squared for mesons:

$$m^2_{n,l} = 8\varkappa\,\beta \left(n + \frac{l}{2} \right) + 2M_0^2, \quad \text{where} \quad \beta^2 = \frac{2\,g^2}{3\,\varkappa}, \qquad (10)$$

n and l are radial and orbital quantum numbers respectively.

Similarly for the case of baryons:

$$m^2_{n,\,l,\,\widetilde{n},\,\widetilde{l}} = 8\varkappa\,\beta\left(n + \frac{l}{2} + \widetilde{n} + \frac{\widetilde{l}}{2}\right) + 3M_0^2, \qquad (11)$$

where $\beta^2 = \frac{2\,g^2}{3\,\varkappa}$.
These spectra being linear and equidistant, qualitatively describe experimental data.

It is also possible to calculate matrix elements $\langle p', \lambda | J^{\mu}(0) | p, \lambda \rangle$ of the electromagnetic current J^{μ} between the lowest states in hadron spectrum, and obtain the corresponding electromagnetic form factors.

The simplest way is to consider the current component $J^{+}(x)$, charge density on the LF:

$$J^{+}(x) = (2\pi)^{-3} \sum_{Q,r} \int d^{[3]}q \, d^{[3]}q' \sum_{j=1}^{3} Q \times \\ \times \left(b^{j+}_{r\,Q}(q) \, b^{j}_{r\,Q}(q') - d^{j+}_{r\,Q}(q) \, d^{j}_{r\,Q}(q') \right),$$

where Q denotes the electric charge of quark.

For the lowest state of meson, the wave function is taken in the following form

$$f_{s_1,s_2,\lambda}(\mathbf{y}) = N \exp\left(-\frac{\beta}{2} \mathbf{y}^2 \right) \langle 1/2, s_1; 1/2, -s_2 | j, \lambda \rangle,$$

where $\langle 1/2, s_1; 1/2, -s_2 | j, \lambda \rangle$ is the Clebsch-Gordan coefficient.

As a result, we obtain:

$$\langle p', \lambda | J^+(0) | p, \lambda \rangle = \frac{2p_- \, Q}{(2\pi)^3} \left(1 + \frac{q^2}{4m^2}\right)^{-1/2} \times$$

$$\times \exp\left(-\frac{q^2}{16\,\beta\,\left(1 + \frac{q^2}{4m^2}\right)}\right) = 2p_- F(q^2),$$

where $F(q^2)$ is the meson form factor:

$$F(q^2) = \frac{Q}{(2\pi)^3} \left(1 + \frac{q^2}{4m^2}\right)^{-1/2} \exp\left(-\frac{q^2}{16\,\beta\,\left(1 + \frac{q^2}{4m^2}\right)}\right). \quad (12)$$

Similar calculations for the lowest baryon state lead to the following expression for baryon form factor:

$$F(q^2) = \frac{Q}{(2\pi)^3}\left(1+\frac{q^2}{4m^2}\right)^{-1}\exp\left(-\frac{q^2}{12\,\beta\,\left(1+\frac{q^2}{4m^2}\right)}\right). \quad (13)$$

These formulas show the difference from the formulae given in the textbook [B.S. Ishkhanov et al.]:

for the lightest mesons $F(q^2) = \left(1\,+\,\frac{q^2}{0{,}52\ \mathrm{GeV}^2}\right)^{-1}$,

for nucleons $F(q^2) = \left(1\,+\,\frac{q^2}{0{,}71\ \mathrm{GeV}^2}\right)^{-2}$.

The asymptotics of these formulas for large q^2, obtained theoretically, differs from ours [S.J. Brodsky et al.; A.V. Efremov and A.V. Radyushkin].

To remove this problem, it is proposed to introduce into the w.f. of state in rest frame the dependence on additional fields E_μ and $\widetilde{E}_\mu$ in the form of factors $\Phi(\mathbf{E})\,\widetilde{\Phi}(\widetilde{\mathbf{E}})$ which are spectral functions for harmonic oscillators in $\mathbf{E}$ and $\widetilde{\mathbf{E}}$.
Then the expression for P^2 should be changed as follows:

$$P^2 = -\frac{g^2}{3}(\nu+1)\left(\triangle_B + \frac{4}{3}\triangle_{\widetilde{B}}\right) + \varkappa\sum\int d^{\,[3]}q\Big\{ b_s^{i+}(q)\, b_s^i(q)\times$$

$$\times\,(\mathbf{u}+\mathbf{B}_i)^2 + d_s^{i+}(q)\, d_s^i(q)\,(\mathbf{u}-\mathbf{B}_i)^2\Big\} + \nu M_0^2 - \frac{\nu}{3}(\nu+1)\sqrt{6\varkappa}g +$$

$$+\left(-g_1^2\triangle_E + \mathbf{E}^2 - 3g_1\right) + (\nu-2)^2\left(-\widetilde{g}_1^2\triangle_{\widetilde{E}} + \widetilde{\mathbf{E}}^2 - 3\widetilde{g}_1\right),$$

$$m^2_{n,\,l,\,\widetilde{n},\,\widetilde{l},\,n_1,\,l_1,\,\widetilde{n}_1,\,\widetilde{l}_1} = 8\varkappa\beta\left(n+\frac{l}{2}+(\nu-2)\left(\widetilde{n}+\frac{\widetilde{l}}{2}\right)\right)+\nu M_0^2+$$

$$+4g_1\left(n_1+\frac{l_1}{2}\right)+4\widetilde{g}_1\,(\nu-2)\left(\widetilde{n}_1+\frac{\widetilde{l}_1}{2}\right). \qquad (15)$$

However we consider only large values of the new parameters g_1, $\widetilde{g}_1$ (g_1/g, $\widetilde{g}_1/g \gg 1$). So it is possible to discard the specified contribution and consider $\Phi(\mathbf{E})\,\widetilde{\Phi}(\widetilde{\mathbf{E}}) = \exp\left(-\frac{1}{2g_1}\mathbf{E}^2-\frac{1}{2\widetilde{g}_1}\widetilde{\mathbf{E}}^2\right)$, which corresponds to the lowest state in new fields.

Then as the result of calculation, we obtain the meson form factor:

$$F(q^2) = \frac{Q}{(2\pi)^3}\left(1+\frac{q^2}{4m^2}\right)^{-1}\exp\left(-\frac{q^2}{16\,\beta\left(1+\frac{q^2}{4m^2}\right)}\right). \qquad (16)$$

Similarly for the baryon:

$$F(q^2) = \frac{Q}{(2\pi)^3}\left(1+\frac{q^2}{4m^2}\right)^{-2}\exp\left(-\frac{q^2}{12\,\beta\left(1+\frac{q^2}{4m^2}\right)}\right). \quad (17)$$

Therefore the correct asymptotics is restored. However the formulae still differ from the formulae of B.S. Ishkhanov. It turned out to be difficult to conform these formulae by varying the parameter β, since in our model there is the degeneracy in mass for particles with different spins and zero values of n and l (π-, ρ- and K-mesons). If we take wave functions in the following form:

$$f_{s_1,s_2,\lambda}(\mathbf{y}) = f(\mathbf{y})\,\langle 1/2, s_1; 1/2, -s_2|j,\lambda\rangle,$$

where $\langle 1/2, s_1; 1/2, -s_2|j,\lambda\rangle$ is the Clebsch-Gordan coefficient, then we can introduce an operator that gives on these states different eigenvalues equal to the spin (either j=0 or j=1).

Such operator can be constructed as the square of the following expression:

$$\vec{J} = \int d^{[3]}q \left(b^{i+}(q,s) \left(\frac{\vec{\sigma}}{2} \right)_{ss'} b^{i}(q,s') + \right.$$
$$\left. + \, d^{i+}(q,s) \left(\frac{\vec{\sigma}}{2} \right)_{ss'} d^{i}(q,s') \right)$$

Then the states of mesons with different spins j can be separated in the mass spectrum if we introduce into effective Hamiltonian the dependence on operator $(\vec{J})^2$.

Conclusion:
In this work we investigated the possibility of constructing an effective Hamiltonian on the light front (LF) within the framework of the simplest quark model of hadrons (i.e., the valence quark model).
We also calculated electromagnetic form factors for the lowest states in these spectra and indicated possible way of obtaining the correct asymptotics of these form factors for large values of the transferred momentum.
In addition, we constructed an operator that allows us to distinguish between states with different spins composed of the spins of quarks.

References:

B.S. Ishkhanov et al., textbook, 2007

S.J. Brodsky et al., Phys. Lett. B, vol. 91 (2), p. 239, 1980