Wigner Function Moments Method

E. B. Balbutsev, I. V. Molodtsova

The **Wigner Function Moments (WFM) method**
is an effective tool for studying collective dynamics in atomic nuclei (and any other many body systems). The solution of Time Dependent Hartree-Fock-Bogoliubov (TDHFB) equations by this method allowed us to find the energies and excitation probabilities of giant resonances (isoscalar and isovector quadrupole ones, isoscalar (compressional) and isovector dipole ones [1]) and various low lying modes. The especial interest among the latter represent "nuclear scissors", the theory of which was created with the help of WFM method. The creation of the theory led to the discovery of two new types of scissors modes, which exist only due to spin degrees of freedom (spin scissors) [2].

[1] E. B. Balbutsev, J. Piperova, M. Durand, I.V. Molodtsova, and A. V. Unzhakova, *Nucl. Phys. A* **571** (1994) 413.
[2] E. B. Balbutsev, I.V. Molodtsova, A. V. Sushkov, N. Yu. Shirikova, and P. Schuck, *Phys. Rev. C* **105**, 044323 (2022).

The irreducible tensors of various ranks are used as the collective variables of the WFM method. Analysing the second rank tensors we have discovered the existence of the so called "hidden angular momenta" of an atomic nucleus. Due to this peculiarity of atomic nuclei one can classify them as the antiferromagnets. The antiferromagnetism of atomic nuclei becomes apparent in the phenomenon of energy levels splitting at the zero deformation (Zeeman effect) [3].

[3] E. B. Balbutsev, and I.V. Molodtsova, *Int. J. Phys. E* **33** №12, 2441031 (2024).

The principal feature of the WFM method, which distinguish it from the random phase approximation, is that it works with the dynamical (time dependent) mean field. Due to it there is no necessity to introduce a residual interaction. More of it, one has not any problems with "spurious" states – they don't appear!

It is necessary to emphasize that we don't seek for the exact (or approximate) solution of TDHFB equations. As a matter of fact, we extract the exact information about the dynamics of average values of various operators. The only (and non-avoidable) approximation is the neglect by the coupling with the dynamics of higher rank tensors (moments). The undoubted merit of WFM method is the possibility to study the large amplitude motion, not only the small amplitudes. For example, we have studied multiphonon giant quadrupole and monopole resonances [4].

[4] E. B. Balbutsev, and P. Schuck, *Yad.Fiz.* **60** №5, 855 (1997).

⇒ Time Dependent Hartree-Fock-Bogoliubov equation

$$i\hbar\dot{\mathcal{R}} = [\mathcal{H}, \mathcal{R}], \tag{1}$$

$$\text{with} \quad \mathcal{R} = \begin{pmatrix} \hat{\rho} & -\hat{æ} \\ -\hat{æ}^\dagger & 1-\hat{\rho}^* \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \hat{h} & \hat{\Delta} \\ \hat{\Delta}^\dagger & -\hat{h}^* \end{pmatrix}.$$

⇒ Microscoic Hamiltonian:

$$\hat{h} = \sum_{i=1}^{A}\left[\frac{\hat{\mathbf{p}}_i^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}_i^2 - \eta\hat{\mathbf{l}}_i\hat{\mathbf{S}}_i\right] + H_{qq} + H_{ss},$$

$$H_{qq} = \sum_{\mu=-2}^{2}(-1)^\mu\left\{\bar{\kappa}\sum_i^Z\sum_j^N + \frac{\kappa}{2}\left[\sum_{i,j(i\neq j)}^{Z} + \sum_{i,j(i\neq j)}^{N}\right]\right\}q_{2-\mu}(\mathbf{r}_i)q_{2\mu}(\mathbf{r}_j),$$

$$H_{ss} = \sum_{\mu=-1}^{1}(-1)^\mu\left\{\bar{\chi}\sum_i^Z\sum_j^N + \frac{\chi}{2}\left[\sum_{i,j(i\neq j)}^{Z} + \sum_{i,j(i\neq j)}^{N}\right]\right\}\hat{S}_{-\mu}(i)\hat{S}_\mu(j),$$

where $q_{2\mu}(\mathbf{r}) = \sqrt{16\pi/5}\, r^2 Y_{2\mu}(\theta, \phi)$.

⇒ Fourier (Wigner) transformation

$$f^{\tau\sigma\sigma'}(\mathbf{r}, \mathbf{p}, t) = \int d\mathbf{s}\, e^{-i\mathbf{p}\mathbf{s}/\hbar}\left\langle \mathbf{r} + \frac{\mathbf{s}}{2}, \tau\sigma|\hat{\rho}|\mathbf{r} - \frac{\mathbf{s}}{2}, \tau\sigma'\right\rangle, \quad \sigma\sigma' = \uparrow\uparrow, \downarrow\downarrow, \uparrow\downarrow, \downarrow\uparrow,$$

with the conventional notation $\uparrow$ for $\sigma = \frac{1}{2}$ and $\downarrow$ for $\sigma = -\frac{1}{2}$. τ - isospin.

$\Rightarrow$ Integrating the equations over the phase space with weights:

$$1, \quad \{r\otimes r\}_{\lambda\mu}, \quad \{p\otimes p\}_{\lambda\mu}, \quad \{r\otimes p\}_{\lambda\mu}, \quad \text{where } \{a\otimes b\}_{\lambda\mu} = \sum_{\gamma,\nu} C^{\lambda\mu}_{1\gamma,1\nu} a_\gamma b_\nu,$$

we obtain a system of nonlinear dynamic equations for the following second order moments $\equiv$ collective variables:

$$R^{\tau\sigma\sigma'}_{\lambda\mu}(t) = (2\pi\hbar)^{-3}\int d\mathbf{r}\int d\mathbf{p}\,\{r\otimes r\}_{\lambda\mu} f^{\tau\sigma\sigma'}(\mathbf{r},\mathbf{p},t),$$

$$P^{\tau\sigma\sigma'}_{\lambda\mu}(t) = (2\pi\hbar)^{-3}\int d\mathbf{r}\int d\mathbf{p}\,\{p\otimes p\}_{\lambda\mu} f^{\tau\sigma\sigma'}(\mathbf{r},\mathbf{p},t),$$

$$L^{\tau\sigma\sigma'}_{\lambda\mu}(t) = (2\pi\hbar)^{-3}\int d\mathbf{r}\int d\mathbf{p}\,\{r\otimes p\}_{\lambda\mu} f^{\tau\sigma\sigma'}(\mathbf{r},\mathbf{p},t),$$

$$F^{\tau\sigma\sigma'}(t) = (2\pi\hbar)^{-3}\int d\mathbf{r}\int d\mathbf{p}\, f^{\tau\sigma\sigma'}(\mathbf{r},\mathbf{p},t), \text{ where } \tau \text{ is an isotopic index.}$$

$\Rightarrow$ Isoscalar and isovector variables:

$$X_{\lambda\mu}(t) = X^{\rm n}_{\lambda\mu}(t) + X^{\rm p}_{\lambda\mu}(t), \quad \bar{X}_{\lambda\mu}(t) = X^{\rm n}_{\lambda\mu}(t) - X^{\rm p}_{\lambda\mu}(t).$$

$\Rightarrow$ Spin-scalar and spin-vector variables:

$$X^{+}_{\lambda\mu}(t) = X^{\uparrow\uparrow}_{\lambda\mu}(t) + X^{\downarrow\downarrow}_{\lambda\mu}(t), \quad X^{-}_{\lambda\mu}(t) = X^{\uparrow\uparrow}_{\lambda\mu}(t) - X^{\downarrow\downarrow}_{\lambda\mu}(t), \qquad X = \{R, P, L, F\}.$$

Physical meaning of the collective variables:

$R^{+}_{2\mu}$ – quadrupole moment of the nucleus and R^{+}_{00} – mean square radius,
$P^{+}_{2\mu}$ and P^{+}_{00} – quadrupole moment and mean square radius in a momentum space.
$L^{+}_{1\mu}$ – orbital angular momentum,
$L^{+}_{2\mu} \sim \dot{\mathcal{R}}^{+}_{2\mu}$ represents the velocity of changing of the nuclear shape,
$L^{+}_{00} \sim \dot{\mathcal{R}}^{+}_{00}$ represents the velocity of changing of the nuclear size,
$F^{+} = A$ – number of particles.

To describe the $K^{\pi} = 1^{+}$ states we need a part of dynamical equations with $\mu = 1$.

$\Rightarrow$ Small amplitude approximation: $X^{\zeta}_{\lambda\mu}(t) = X^{\zeta\,\mathrm{eq}}_{\lambda\mu} + \mathcal{X}^{\zeta}_{\lambda\mu}(t)$,

$$\mathcal{X}^{\zeta}_{\lambda\mu}(t) = \{\mathcal{R}^{\zeta}(t), \mathcal{L}^{\zeta}(t), \mathcal{P}^{\zeta}(t), \mathcal{F}^{\zeta}(t)\}, \quad \zeta = +, -, \uparrow\downarrow, \downarrow\uparrow.$$

Imposing the time evolution via $e^{i\Omega t}$ for all variables allows to transform the system of nonlinear dynamical equations into a set of linear algebraic equations.

Eigenfrequencies Ω are found as solutions of its secular equation.

Simple example: $H=\sum_{i=1}^{A}\left(\frac{\mathbf{p}_i^2}{2m}+\frac{1}{2}m\omega^2\mathbf{r}_i^2\right)+H_{qq}$,

Coupled dynamical nonlinear equations for protons ($\tau=$p) and neutrons ($\tau=$n):

$$\frac{d}{dt}R^{\tau}_{\lambda\mu}-\frac{2}{m}L^{\tau}_{\lambda\mu}=0,\quad \lambda=0,2$$

$$\frac{d}{dt}L^{\tau}_{\lambda\mu}-\frac{1}{m}P^{\tau}_{\lambda\mu}+m\,\omega^2R^{\tau}_{\lambda\mu}-12\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\left\{\begin{smallmatrix}1&1&j\\2&\lambda&1\end{smallmatrix}\right\}\{Z_2^{\tau}\otimes R_j^{\tau}\}_{\lambda\mu}=0,\quad \lambda=0,1,2$$

$$\frac{d}{dt}P^{\tau}_{\lambda\mu}+2m\,\omega^2L^{\tau}_{\lambda\mu}-24\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\left\{\begin{smallmatrix}1&1&j\\2&\lambda&1\end{smallmatrix}\right\}\{Z_2^{\tau}\otimes L_j^{\tau}\}_{\lambda\mu}=0,\quad \lambda=0,2$$

where $\left\{\begin{smallmatrix}1&1&j\\2&\lambda&1\end{smallmatrix}\right\}$ is the Wigner 6j-symbol, $\{Z_2^{\tau}\otimes X_j^{\tau}\}_{\lambda\mu}=\sum_{\sigma,\nu}C^{\lambda\mu}_{2\sigma,j\nu}Z^{\tau}_{2\sigma}X^{\tau}_{j\nu}$ with

$$Z^{\rm n}_{2\mu}=\kappa R^{\rm n}_{2\mu}+\bar{\kappa}R^{\rm p}_{2\mu}\,,\quad Z^{\rm p}_{2\mu}=\kappa R^{\rm p}_{2\mu}+\bar{\kappa}R^{\rm n}_{2\mu}\,.$$

E. Balbutsev and P. Schuck, NPA **720** (2003) 293.

$R^{\rm eq}_{00}=-Q_{00}/\sqrt{3},\quad R^{\rm eq}_{20}=Q_{20}/\sqrt{6},$
$Q_{00}=\frac{3}{5}AR^2,\quad Q_{20}=\frac{4}{3}\delta Q_{00},\quad \delta$ – deformation parameter, $\quad Q^{\tau}_{00}=(N^{\tau}/A)Q_{00}$.

Simple example: $H = \sum_{i=1}^{A} \left(\frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} m\omega^2 \mathbf{r}_i^2 \right) + H_{qq},$

Coupled dynamical nonlinear equations for protons ($\tau =$p) and neutrons ($\tau =$n):

$$\frac{d}{dt} R^\tau_{\lambda\mu} - \frac{2}{m} L^\tau_{\lambda\mu} = 0, \quad \lambda = 0, 2$$

$$\frac{d}{dt} L^\tau_{\lambda\mu} - \frac{1}{m} P^\tau_{\lambda\mu} + m\,\omega^2 R^\tau_{\lambda\mu} - 12\sqrt{5} \sum_{j=0}^{2} \sqrt{2j+1} \left\{ \begin{smallmatrix} 1 & 1 & j \\ 2 & \lambda & 1 \end{smallmatrix} \right\} \{Z_2^\tau \otimes R_j^\tau\}_{\lambda\mu} = 0, \quad \lambda = 0, 1, 2$$

$$\frac{d}{dt} P^\tau_{\lambda\mu} + 2m\,\omega^2 L^\tau_{\lambda\mu} - 24\sqrt{5} \sum_{j=0}^{2} \sqrt{2j+1} \left\{ \begin{smallmatrix} 1 & 1 & j \\ 2 & \lambda & 1 \end{smallmatrix} \right\} \{Z_2^\tau \otimes L_j^\tau\}_{\lambda\mu} = 0, \quad \lambda = 0, 2$$

where $\left\{ \begin{smallmatrix} 1 & 1 & j \\ 2 & \lambda & 1 \end{smallmatrix} \right\}$ is the Wigner 6j-symbol, $\{Z_2^\tau \otimes X_j^\tau\}_{\lambda\mu} = \sum_{\sigma,\nu} C^{\lambda\mu}_{2\sigma,j\nu} Z^\tau_{2\sigma} X^\tau_{j\nu}$ with

$$Z^{\mathrm{n}}_{2\mu} = \kappa R^{\mathrm{n}}_{2\mu} + \bar{\kappa} R^{\mathrm{p}}_{2\mu}\,, \quad Z^{\mathrm{p}}_{2\mu} = \kappa R^{\mathrm{p}}_{2\mu} + \bar{\kappa} R^{\mathrm{n}}_{2\mu}\,.$$

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$R^{\mathrm{eq}}_{00} = -Q_{00}/\sqrt{3}, \quad R^{\mathrm{eq}}_{20} = Q_{20}/\sqrt{6},$
$Q_{00} = \frac{3}{5} A R^2, \quad Q_{20} = \frac{4}{3}\delta Q_{00}, \quad \delta$ – deformation parameter, $\quad Q^\tau_{00} = (N^\tau/A) Q_{00}.$

For TDHFB (with spin) we obtain a system of 44 coupled isovector and isoscalar dynamical equations.

Eigenvalues

^{164}Dy

Decoupled equations $\Rightarrow$

E_i (MeV)	$B(M1)_i$ (μ_N^2)	$B(E2)_i$ ($W.u.$)
1.29	0.01	**53.25**
2.62	0.09	2.91
2.44	**2.03**	0.34
3.35	**1.36**	1.62
10.94	0.00	**55.12**
14.04	0.00	2.78
14.60	0.06	0.48
15.88	0.00	0.55
16.46	0.07	0.36
17.69	0.00	0.45
17.90	0.00	0.51
18.22	0.18	1.85
19.32	0.10	0.97
21.29	2.47	**31.38**

Coupled equations

E_i (MeV)	$B(M1)_i$ (μ_N^2)	$B(E2)_i$ ($W.u.$)
1.47	0.05	**25.68**
2.20	**1.76**	3.30
2.87	**2.24**	0.34
3.59	**1.56**	4.37
10.92	0.04	**50.37**
13.10	0.00	2.85
15.42	0.07	0.57
15.55	0.00	1.12
16.78	0.06	0.53
17.69	0.01	0.68
17.91	0.00	0.53
18.22	0.13	0.89
19.32	0.08	0.61
21.26	2.03	**21.60**

3 **magnetic** states correspond to 3 physically possible **Scissors modes**:

- **(a)** spin-scalar isovector (conventional, orbital scissors)
- **(b)** spin-vector isoscalar (spin scissors)
- **(c)** spin-vector isovector (spin scissors)

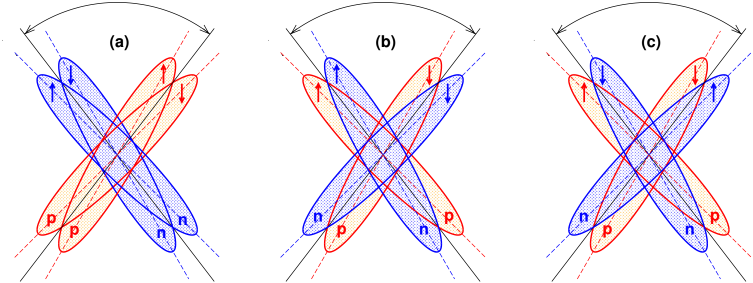

Figure: Schematic representation of three interconnected scissors modes. Arrows show the direction of spin projections; p – protons, n – neutrons.

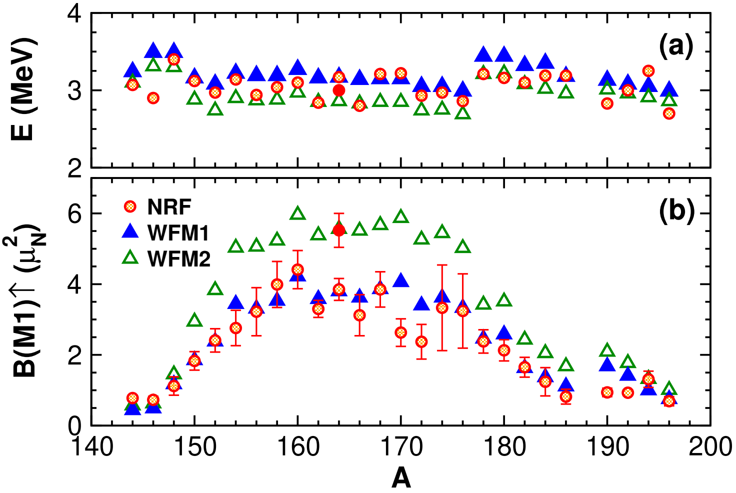

Figure: WFM1 – the sum of two highest scissors, WFM2 – the sum of three scissors.

Electric 1^+ state below nuclear scissors

The nature of the lowest state can be understood after solving dynamical equations with $\mu = 2$ and $\mu = 0$, and studying the deformation dependence.

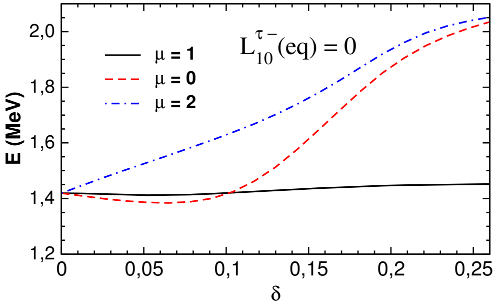

Figure: Energies E of the lowest electrical $\mu = 0, 1, 2$ levels as a function of deformation δ. The calculations were performed without $L_{10}^{\tau-}(\mathrm{eq})$.

Electric 1^+ state below nuclear scissors

The phenomenon "hidden angular momenta" [BMS, PRC **91**, 064312 (2015)].

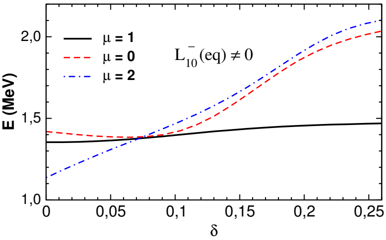

Figure: Energy branches ($\mu = 0, 1, 2$) vs. deformation.

$$L^{\zeta}_{\lambda\mu} = \int d\mathbf{r} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \{r \otimes p\}_{\lambda\mu} f^{\zeta}(\mathbf{r}, \mathbf{p})$$

$$\mu_z = \frac{1}{2} \int d\mathbf{r}\, [\mathbf{r} \times \mathbf{j}]_z = -\frac{i}{\hbar\sqrt{2}} L_{10}$$

$$L^-_{10} = L^{\uparrow\uparrow}_{10} - L^{\downarrow\downarrow}_{10}$$

$L^{\uparrow\uparrow}_{10}$ and $L^{\downarrow\downarrow}_{10}$ are the average values of the z-component of the orbital angular momentum of all nucleons with the spin projections $\uparrow$ and $\downarrow$.

In the equilibrium state: $L^{\downarrow\downarrow}_{10}(\mathrm{eq}) = -L^{\uparrow\uparrow}_{10}(\mathrm{eq})$

$$L^-_{10}(\mathrm{eq}) = i\hbar\frac{\sqrt{2}}{6}\eta m\left[\left(1 - \frac{3}{8}\frac{\eta^2\hbar^2}{\omega^2}\right) + \frac{4}{3}\delta\right] Q_{00}$$

$$L^+_{10}(\mathrm{eq}) = L^{\uparrow\uparrow}_{10}(\mathrm{eq}) + L^{\downarrow\downarrow}_{10}(\mathrm{eq}) = 0$$

The ground-state nucleus consists of two equal parts having nonzero angular momenta with opposite directions, which compensate each other resulting in the zero total angular momentum whereas $L^-_{10}(\mathrm{eq}) \neq 0$ *even in the spherical limit*.

$\longrightarrow$ **Antiferromagnetism**

The influence of an external magnetic field

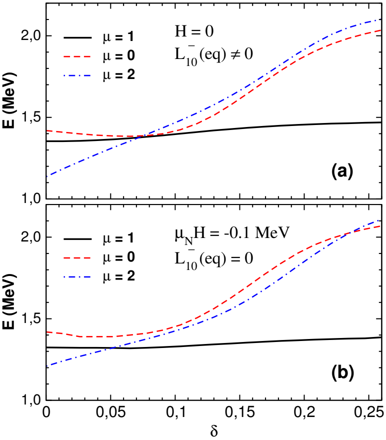

Figure (b): for zero deformation, the energy of splitting (Zeeman energy) is $\Delta E \simeq \mu \mathcal{H} \mu_N$ MeV.

The action of an uniform magnetic field $\mathcal{H}$ can be described by adding to the Hamiltonian the term

$$\mathcal{M} = -\hat{\mu} \cdot \hat{\mathcal{H}},$$

$$\text{where} \quad \hat{\mu} = \frac{e}{2mc} \sum_{\tau=\mathrm{p,n}} \left(g_l^\tau \hat{\mathbf{l}}^\tau + g_s^\tau \hat{\mathbf{s}}^\tau \right)$$

– operator of the magnetic moment of the nucleus.

In the case $\hat{\mathcal{H}} = \mathcal{H}_z \equiv \mathcal{H}$:

$$\mathcal{M}^\tau = -\frac{e\mathcal{H}}{2mc}(g_l^\tau \hat{l}_z + g_s^\tau \hat{s}_z).$$

The Wigner transformation of $\mathcal{M}^\tau$:

$$\mathcal{M}_W^\tau = \mu_N \mathcal{H} \left(g_l^\tau \frac{i}{\hbar} \sqrt{2} \{r \otimes p\}_{10} - g_s^\tau \frac{1}{2} \sigma_0 \right).$$

Table: Splitting for $\delta = 0$:

I: $\mathcal{H} = 0$, $L^-_{10}(\mathrm{eq}) = 22i\hbar$;

$L^-_{10}(\mathrm{eq}) = 0$:
- II: $\mu_N \mathcal{H} = -0.13$ MeV,
- III: $\mu_N \mathcal{H} = -0.10$ MeV,
- IV: $\mu_N \mathcal{H} = -0.05$ MeV.

μ	E, MeV			
	I	II	III	IV
0	1.42	1.42	1.42	1.42
1	1.35	1.28	1.32	1.37
2	1.14	1.15	1.21	1.31

Currents

$$\mathbf{j}^{\tau\varsigma}(\mathbf{r},t)=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\,\mathbf{p}\,\delta f^{\tau\varsigma}(\mathbf{r},\mathbf{p},t). \tag{2}$$

$$\mathrm{j}_i^{\tau\varsigma}(\mathbf{r},t)=n(\mathbf{r})\left[K_i^{\tau\varsigma}(t)+\sum_j(-1)^j K_{i,-j}^{\tau\varsigma}(t)r_j+\sum_{\lambda',\mu'}(-1)^{\mu'}K_{i,\lambda',-\mu'}^{\tau\varsigma}(t)\{r\otimes r\}_{\lambda'\mu'}+\ldots\right],$$

where ς – spin index, $n(\mathbf{r})$ – nuclear density.
The coefficients $K_{i,-j}^{\tau\varsigma}(t)$ are connected by linear relations with the collective variables $\mathcal{L}_{\lambda\mu}^{\tau\varsigma}(t)$:

$$\mathcal{L}_{\lambda\mu}^{\tau\varsigma}(t)=\int d\mathbf{r}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\{r\otimes p\}_{\lambda\mu}\delta f^{\tau\varsigma}(\mathbf{r},\mathbf{p},t)=\int d\mathbf{r}\,\{r\otimes \mathrm{j}^{\tau\varsigma}(\mathbf{r},t)\}_{\lambda\mu}.$$

We find in Cartesian coordinates:

$$\mathrm{j}_x^{\tau+}=0,$$
$$\mathrm{j}_y^{\tau+}=i\alpha_1 n(\mathbf{r})(\mathcal{L}_{11}^{\tau+}-\mathcal{L}_{21}^{\tau+})z,$$
$$\mathrm{j}_z^{\tau+}=i\alpha_2 n(\mathbf{r})(\mathcal{L}_{11}^{\tau+}+\mathcal{L}_{21}^{\tau+})y.$$

This result is quite remarkable. The equation $\delta J_x^{\varsigma}=0$ says that all motions take place only in two dimensions ($\mathcal{J}_x^{\tau+}=0$), i.e. in one plane, as it should be for real scissors.

✓ Displacement field is a superposition of rotational and irrotational flows.
✓ Displacement pattern will be determined by the competition between these two contributions.

Scissors currents

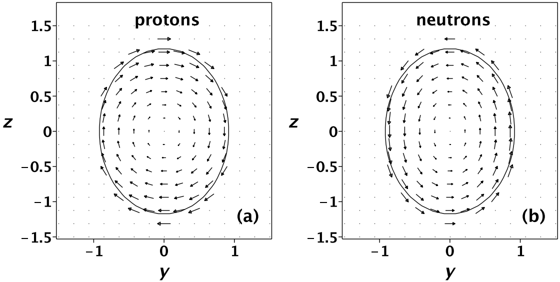

Figure: The proton and neutron currents in ^{164}Dy for scissors state with energy $E = 3.59$ MeV (conventional scissors).

Electric 1^+ state below nuclear scissors: Currents

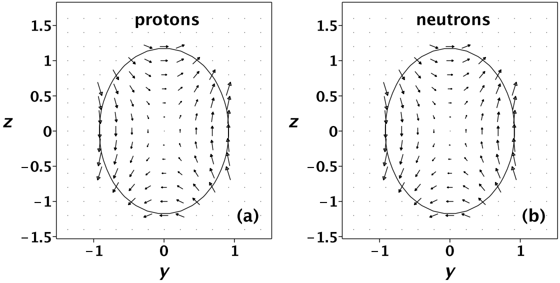

Figure: The proton and neutron currents in ^{164}Dy for the lowest electrical level with energy $E = 1.29$ MeV. The **calculations** were performed **without the coupling of isovector and isoscalar equations**

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Electric 1^+ state below nuclear scissors: Currents

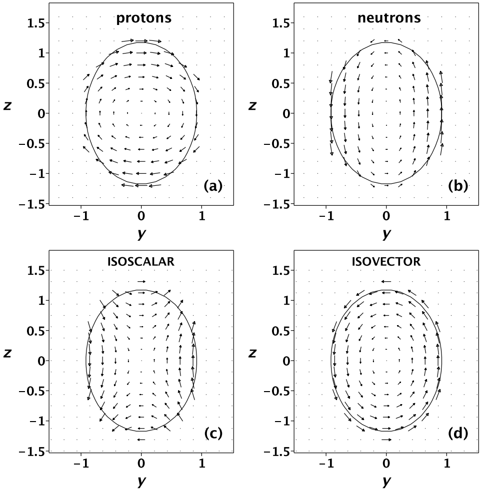


Figure: The currents in ^{164}Dy for the lowest (electrical) level with energy $E = 1.47$ MeV.
The isoscalar-isovector coupling is taken into account.

Electric 1^+ state below nuclear scissors: Currents

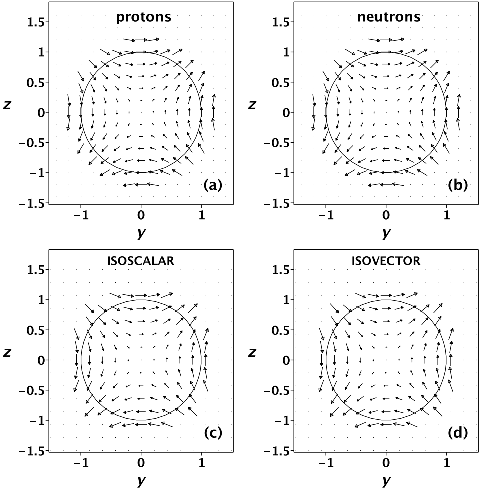
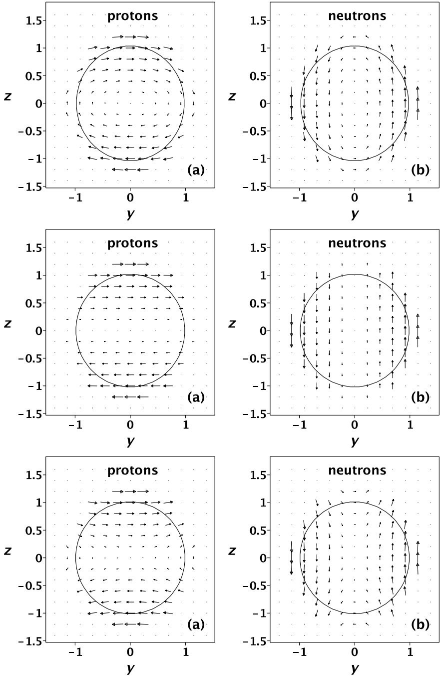


Figure: The currents **in spherical** nucleus with N and Z corresponding to ^{164}Dy for the lowest (electrical) level.

Nucleons currents in the nucleus with N and Z corresponding to ^{164}Dy

for:



$\delta = 0.06$

$\delta = 0.032$

$\delta = 0.02$

THANK YOU

Electric 1^+ state below nuclear scissors: Currents

The total angular momentum $\langle\hat{\mathbf{J}}\rangle = \langle\hat{\mathbf{l}}\rangle + \langle\hat{\mathbf{s}}\rangle$ can be written in terms of dynamical variables:

$$\langle\hat{J}_1\rangle = -i\sqrt{2}L^+_{11} - \frac{\hbar}{\sqrt{2}}F^{\downarrow\uparrow}, \quad \langle\hat{J}_0\rangle = -i\sqrt{2}L^+_{10} + \frac{\hbar}{2}F^-, \quad \langle\hat{J}_{-1}\rangle = -i\sqrt{2}L^+_{1-1} + \frac{\hbar}{\sqrt{2}}F^{\uparrow\downarrow}.$$

$$\frac{d}{dt}\mathcal{J}_1(t) = -\sqrt{2}\frac{d}{dt}\left[i\mathcal{L}^+_{11}(t) + \frac{\hbar}{2}\mathcal{F}^{\downarrow\uparrow}(t)\right] = 0 \quad \longrightarrow \quad \boxed{\mathcal{J}_1 = const = 0}$$

$$\frac{d}{dt}\bar{\mathcal{J}}_1(t) = -\sqrt{2}\frac{d}{dt}\left[i\bar{\mathcal{L}}^+_{11}(t) + \frac{\hbar}{2}\bar{\mathcal{F}}^{\downarrow\uparrow}(t)\right] = -i\,3\sqrt{2}\,\delta m\omega^2\left[\bar{\mathcal{R}}^+_{21}(t) - \frac{(N-Z)}{A}\mathcal{R}^+_{21}(t)\right].$$

$$\bar{\mathcal{J}}_1 = -\sqrt{2}\left[i\bar{\mathcal{L}}^+_{11} + \frac{\hbar}{2}\bar{\mathcal{F}}^{\downarrow\uparrow}\right] = -6\sqrt{2}\,\delta m\frac{\omega^2}{\Omega}\frac{NZ}{A}\left[\frac{\mathcal{R}^{\mathrm{n}+}_{21}}{N} - \frac{\mathcal{R}^{\mathrm{p}+}_{21}}{Z}\right]$$

Electric 1^+ state below nuclear scissors: Currents

The total angular momentum $\langle\hat{\mathbf{J}}\rangle = \langle\hat{\mathbf{l}}\rangle + \langle\hat{\mathbf{s}}\rangle$ can be written in terms of dynamical variables:

$$\langle\hat{J}_1\rangle = -i\sqrt{2}L^+_{11} - \frac{\hbar}{\sqrt{2}}F^{\downarrow\uparrow}, \quad \langle\hat{J}_0\rangle = -i\sqrt{2}L^+_{10} + \frac{\hbar}{2}F^{-}, \quad \langle\hat{J}_{-1}\rangle = -i\sqrt{2}L^+_{1-1} + \frac{\hbar}{\sqrt{2}}F^{\uparrow\downarrow}.$$

$$\frac{d}{dt}\mathcal{J}_1(t) = -\sqrt{2}\frac{d}{dt}\left[i\mathcal{L}^+_{11}(t) + \frac{\hbar}{2}\mathcal{F}^{\downarrow\uparrow}(t)\right] = 0 \quad \longrightarrow \quad \boxed{\mathcal{J}_1 = const = 0}$$

$$\frac{d}{dt}\bar{\mathcal{J}}_1(t) = -\sqrt{2}\frac{d}{dt}\left[i\bar{\mathcal{L}}^+_{11}(t) + \frac{\hbar}{2}\bar{\mathcal{F}}^{\downarrow\uparrow}(t)\right] = -i\,3\sqrt{2}\,\delta m\omega^2\left[\bar{\mathcal{R}}^+_{21}(t) - \frac{(N-Z)}{A}\mathcal{R}^+_{21}(t)\right].$$

$$\bar{\mathcal{J}}_1 = -\sqrt{2}\left[i\bar{\mathcal{L}}^+_{11} + \frac{\hbar}{2}\bar{\mathcal{F}}^{\downarrow\uparrow}\right] = -6\sqrt{2}\,\delta m\frac{\omega^2}{\Omega}\frac{NZ}{A}\left[\frac{\mathcal{R}^{\rm n+}_{21}}{N} - \frac{\mathcal{R}^{\rm p+}_{21}}{Z}\right]$$

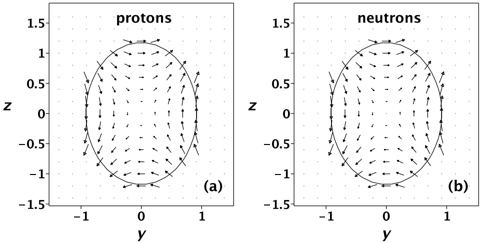


Figure: The proton and neutron currents in ^{164}Dy for the $E = 1.47$ MeV.
The **calculations without** taking into account the **contribution of $\mathcal{L}^+_{11}$, $\bar{\mathcal{L}}^+_{11}$ ($\lambda = 1$) variables**.

The basis of our method is the TDHFB equation in matrix formulation:

$$i\hbar\dot{\mathcal{R}} = [\mathcal{H}, \mathcal{R}] \tag{3}$$

with

$$\mathcal{R} = \begin{pmatrix} \hat{\rho} & -\hat{æ} \\ -\hat{æ}^{\dagger} & 1-\hat{\rho}^{*} \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \hat{h} & \hat{\Delta} \\ \hat{\Delta}^{\dagger} & -\hat{h}^{*} \end{pmatrix} \tag{4}$$

The normal density matrix $\hat{\rho}$ and Hamiltonian $\hat{h}$ are hermitian whereas the abnormal density $\hat{æ}$ and the pairing gap $\hat{\Delta}$ are skew symmetric: $\hat{æ}^{\dagger} = -\hat{æ}^{*}$, $\hat{\Delta}^{\dagger} = -\hat{\Delta}^{*}$.
The detailed form of the TDHFB equations is

$$\begin{aligned} i\hbar\dot{\hat{\rho}} &= \hat{h}\hat{\rho} - \hat{\rho}\hat{h} - \hat{\Delta}\hat{æ}^{\dagger} + \hat{æ}\hat{\Delta}^{\dagger}, \\ -i\hbar\dot{\hat{\rho}}^{*} &= \hat{h}^{*}\hat{\rho}^{*} - \hat{\rho}^{*}\hat{h}^{*} - \hat{\Delta}^{\dagger}\hat{æ} + \hat{æ}^{\dagger}\hat{\Delta}, \\ -i\hbar\dot{\hat{æ}} &= -\hat{h}\hat{æ} - \hat{æ}\hat{h}^{*} + \hat{\Delta} - \hat{\Delta}\hat{\rho}^{*} - \hat{\rho}\hat{\Delta}, \\ -i\hbar\dot{\hat{æ}}^{\dagger} &= \hat{h}^{*}\hat{æ}^{\dagger} + \hat{æ}^{\dagger}\hat{h} - \hat{\Delta}^{\dagger} + \hat{\Delta}^{\dagger}\hat{\rho} + \hat{\rho}^{*}\hat{\Delta}^{\dagger}. \end{aligned} \tag{5}$$

Let us consider matrix form of (5) in coordinate space keeping spin indices s, s' with compact notation $X_{rr'}^{ss'} \equiv \langle \mathbf{r}, s|\hat{X}|\mathbf{r}', s'\rangle$. Then the set of TDHFB equations with specified spin indices reads:

$$
\begin{aligned}
i\hbar\dot{\rho}_{rr''}^{\uparrow\uparrow} &= \int d^3r' (h_{rr'}^{\uparrow\uparrow}\rho_{r'r''}^{\uparrow\uparrow} - \rho_{rr'}^{\uparrow\uparrow}h_{r'r''}^{\uparrow\uparrow} + \hat{h}_{rr'}^{\uparrow\downarrow}\rho_{r'r''}^{\downarrow\uparrow} - \rho_{rr'}^{\uparrow\downarrow}h_{r'r''}^{\downarrow\uparrow} - \Delta_{rr'}^{\uparrow\downarrow}æ^{\dagger}{}_{r'r''}^{\downarrow\uparrow} + æ_{rr'}^{\uparrow\downarrow}\Delta^{\dagger}{}_{r'r''}^{\downarrow\uparrow}),\\
i\hbar\dot{\rho}_{rr''}^{\uparrow\downarrow} &= \int d^3r' (h_{rr'}^{\uparrow\uparrow}\rho_{r'r''}^{\uparrow\downarrow} - \rho_{rr'}^{\uparrow\uparrow}h_{r'r''}^{\uparrow\downarrow} + \hat{h}_{rr'}^{\uparrow\downarrow}\rho_{r'r''}^{\downarrow\downarrow} - \rho_{rr'}^{\uparrow\downarrow}h_{r'r''}^{\downarrow\downarrow}),\\
i\hbar\dot{\rho}_{rr''}^{\downarrow\uparrow} &= \int d^3r' (h_{rr'}^{\downarrow\uparrow}\rho_{r'r''}^{\uparrow\uparrow} - \rho_{rr'}^{\downarrow\uparrow}h_{r'r''}^{\uparrow\uparrow} + \hat{h}_{rr'}^{\downarrow\downarrow}\rho_{r'r''}^{\downarrow\uparrow} - \rho_{rr'}^{\downarrow\downarrow}h_{r'r''}^{\downarrow\uparrow}),\\
i\hbar\dot{\rho}_{rr''}^{\downarrow\downarrow} &= \int d^3r' (h_{rr'}^{\downarrow\uparrow}\rho_{r'r''}^{\uparrow\downarrow} - \rho_{rr'}^{\downarrow\uparrow}h_{r'r''}^{\uparrow\downarrow} + \hat{h}_{rr'}^{\downarrow\downarrow}\rho_{r'r''}^{\downarrow\downarrow} - \rho_{rr'}^{\downarrow\downarrow}h_{r'r''}^{\downarrow\downarrow} - \Delta_{rr'}^{\downarrow\uparrow}æ^{\dagger}{}_{r'r''}^{\uparrow\downarrow} + æ_{rr'}^{\downarrow\uparrow}\Delta^{\dagger}{}_{r'r''}^{\uparrow\downarrow}),\\
i\hbar\dot{æ}_{rr''}^{\uparrow\downarrow} &= -\hat{\Delta}_{rr''}^{\uparrow\downarrow} + \int d^3r' \left(h_{rr'}^{\uparrow\uparrow}æ_{r'r''}^{\uparrow\downarrow} + æ_{rr'}^{\uparrow\downarrow}h^*{}_{r'r''}^{\downarrow\downarrow} + \Delta_{rr'}^{\uparrow\downarrow}\rho^*{}_{r'r''}^{\downarrow\downarrow} + \rho_{rr'}^{\uparrow\uparrow}\Delta_{r'r''}^{\uparrow\downarrow}\right),\\
i\hbar\dot{æ}_{rr''}^{\downarrow\uparrow} &= -\hat{\Delta}_{rr''}^{\downarrow\uparrow} + \int d^3r' \left(h_{rr'}^{\downarrow\downarrow}æ_{r'r''}^{\downarrow\uparrow} + æ_{rr'}^{\downarrow\uparrow}h^*{}_{r'r''}^{\uparrow\uparrow} + \Delta_{rr'}^{\downarrow\uparrow}\rho^*{}_{r'r''}^{\uparrow\uparrow} + \rho_{rr'}^{\downarrow\downarrow}\Delta_{r'r''}^{\downarrow\uparrow}\right).
\end{aligned}
\tag{6}
$$

This set of equations must be complemented by the complex conjugated equations.

Pair potentialn

The Wigner transform of the pair potential (pairing gap) $\Delta(\mathbf{r},\mathbf{p})$ is related to the Wigner transform of the anomalous density by

$$\Delta(\mathbf{r},\mathbf{p}) = -\int \frac{d^3p'}{(2\pi\hbar)^3} v(|\mathbf{p}-\mathbf{p}'|)æ(\mathbf{r},\mathbf{p}'), \qquad (7)$$

where $v(p)$ is a Fourier transform of the two-body interaction. We take for the pairing interaction a simple Gaussian, $v(p) = \beta \mathrm{e}^{-\alpha p^2}$ with $\beta = -|V_0|(r_p\sqrt{\pi})^3$ and $\alpha = r_p^2/4\hbar^2$. The following values of parameters were used in calculations: $r_p = 1.9$ fm, $|V_0| = 25$ MeV. Several exceptions were done for rare earth nuclei: $|V_0| = 26$ MeV for ^{150}Nd, $|V_0| = 26.5$ MeV for ^{176,178,180}Hf and ^{182,184}W, $|V_0| = 27$ MeV for nuclei with deformation $\delta \leqslant 0.18$.

Excitation probabilities

Excitation probabilities are calculated with the help of the theory of linear response of the system to a weak external field

$$\hat{O}(t) = \hat{O}\,\mathrm{e}^{-i\Omega t} + \hat{O}^{\dagger}\, e^{i\Omega t}. \tag{8}$$

The matrix elements of the operator $\hat{O}$ obey the relationship

$$|\langle\psi_a|\hat{O}|\psi_0\rangle|^2 = \hbar \lim_{\Omega\to\Omega_a}(\Omega-\Omega_a)\overline{\langle\psi'|\hat{O}|\psi'\rangle \mathrm{e}^{-i\Omega t}}, \tag{9}$$

where ψ_0 and ψ_a are the stationary wave functions of the unperturbed ground and excited states; ψ' is the wave function of the perturbed ground state,
$\Omega_a = (E_a - E_0)/\hbar$ are the normal frequencies, the bar means averaging over a time interval much larger than $1/\Omega$.
To calculate the magnetic transition probability, it is necessary to excite the system by the following external field:

$$\hat{O}_{\lambda\mu} = \mu_N\left(g_s\hat{\mathbf{S}}/\hbar - ig_l\frac{2}{\lambda+1}[\mathbf{r}\times\nabla]\right)\nabla(r^{\lambda}Y_{\lambda\mu}), \quad \mu_N = \frac{e\hbar}{2mc}. \tag{10}$$

Here $g_l^{\mathrm{p}} = 1$, $g_s^{\mathrm{p}} = 5.5856$ for protons and $g_l^{\mathrm{n}} = 0$, $g_s^{\mathrm{n}} = -3.8263$ for neutrons.

The dipole operator ($\lambda = 1,\ \mu = 1$) in cyclic coordinates looks like

$$\hat{O}_{11} = \mu_N \sqrt{\frac{3}{4\pi}} \left[g_s \hat{S}_1/\hbar - g_l \sqrt{2} \sum_{\nu,\sigma} C^{11}_{1\nu,1\sigma} r_\nu \nabla_\sigma \right]. \quad (11)$$

Its Wigner transform is

$$(\hat{O}_{11})_W = \sqrt{\frac{3}{4\pi}} \left[g_s \hat{S}_1 - i g_l \sqrt{2} \sum_{\nu,\sigma} C^{11}_{1\nu,1\sigma} r_\nu p_\sigma \right] \frac{\mu_N}{\hbar}. \quad (12)$$

For the matrix element we have

$$\begin{aligned}
\langle \psi' | \hat{O}_{11} | \psi' \rangle &= \sqrt{\frac{3}{2\pi}} \left[-\frac{\hbar}{2} (g_s^{\mathrm{n}} \mathcal{F}^{\mathrm{n}\downarrow\uparrow} + g_s^{\mathrm{p}} \mathcal{F}^{\mathrm{p}\downarrow\uparrow}) - i g_l^{\mathrm{p}} \mathcal{L}^{\mathrm{p}+}_{11} \right] \frac{\mu_N}{\hbar} \\
&= \sqrt{\frac{3}{8\pi}} \left[-\frac{1}{2} [(g_s^{\mathrm{n}} - g_s^{\mathrm{p}}) \bar{\mathcal{F}}^{\downarrow\uparrow} + (g_s^{\mathrm{n}} + g_s^{\mathrm{p}}) \mathcal{F}^{\downarrow\uparrow}] - \frac{i}{\hbar} g_l^{\mathrm{p}} (\mathcal{L}^{+}_{11} - \bar{\mathcal{L}}^{+}_{11}) \right] \mu_N \\
&= \sqrt{\frac{3}{8\pi}} \left[\frac{1}{2} (g_s^{\mathrm{p}} - g_s^{\mathrm{n}}) \bar{\mathcal{F}}^{\downarrow\uparrow} + \frac{i}{\hbar} g_l^{\mathrm{p}} \bar{\mathcal{L}}^{+}_{11} + \frac{i}{\hbar} [g_s^{\mathrm{n}} + g_s^{\mathrm{p}} - g_l^{\mathrm{p}}] \mathcal{L}^{+}_{11} \right] \mu_N, \quad (13)
\end{aligned}$$

$$\mathcal{F}^{\downarrow\uparrow}(t) = \int d(\mathbf{p}, \mathbf{r}) \delta f^{\downarrow\uparrow}_{\tau}(\mathbf{r}, \mathbf{p}, t).$$

Deriving (13) we have used the relation $2i\mathcal{L}^+_{11} = -\hbar\mathcal{F}^{\downarrow\uparrow}$, which follows from the angular momentum conservation.
One has to add the external field (11) to the Hamiltonian. Due to the external field some dynamical equations of become inhomogeneous:

$$\dot{\mathcal{R}}^+_{21} = \ldots + i\frac{3}{\sqrt{\pi}}\frac{\mu_N}{4\hbar}g_l^{\rm p}R^+_{20}(\mathrm{eq})\,\mathrm{e}^{i\Omega t},$$
$$\dot{\mathcal{L}}^-_{11} = \ldots + i\sqrt{\frac{3}{\pi}}\frac{\mu_N}{4\hbar}g_l^{\rm p}L^-_{10}(\mathrm{eq})\,\mathrm{e}^{i\Omega t},$$
$$\dot{\mathcal{L}}^{\downarrow\uparrow}_{10} = \ldots + i\sqrt{\frac{3}{2\pi}}\frac{\mu_N}{4\hbar}\,(g_s^{\rm n} - g_s^{\rm p})\,L^-_{10}(\mathrm{eq})\,\mathrm{e}^{i\Omega t}. \qquad (14)$$

Solving the inhomogeneous set of equations one can find the required in (13) values of $\mathcal{L}^+_{11}$, $\bar{\mathcal{L}}^+_{11}$ and $\bar{\mathcal{F}}^{\downarrow\uparrow}$ and using (9) calculate $B(M1)$ factors for all excitations.

To calculate the electric transition probability, it is necessary to excite the system by the external field operator

$$\hat{O}_{2\mu} = er^2 Y_{2\mu} = \beta\{r \otimes r\}_{2\mu}, \tag{15}$$

where $\beta = e\sqrt{\frac{15}{8\pi}}$. The matrix element is given by

$$\langle\psi'|\hat{O}_{2\mu}|\psi'\rangle = \beta\,\mathcal{R}^{\mathrm{p}+}_{2\mu} = \frac{\beta}{2}(\mathcal{R}^{+}_{2\mu} - \bar{\mathcal{R}}^{+}_{2\mu}). \tag{16}$$

For $\mu = 1$ the external field makes inhomogeneous only some proton equations:

$$\begin{aligned}
\dot{\mathcal{L}}^{\mathrm{p}+}_{21} &= \ldots - \frac{2}{3}\beta\left(\frac{Q^{\mathrm{p}}_{20}}{4} + Q^{\mathrm{p}}_{00}\right)\mathrm{e}^{i\Omega t}, \\
\dot{\mathcal{L}}^{\mathrm{p}+}_{11} &= \ldots + \frac{\beta}{2}Q^{\mathrm{p}}_{20}\mathrm{e}^{i\Omega t}, \\
\dot{\mathcal{P}}^{\mathrm{p}-}_{21} &= \ldots - \sqrt{2}\beta L^{\mathrm{p}-}_{10}(\mathrm{eq})\mathrm{e}^{i\Omega t}.
\end{aligned} \tag{17}$$

Solving the inhomogeneous set of the coupled isovector and isoscalar equations one can find the values of $\mathcal{R}^{+}_{21}$ and $\bar{\mathcal{R}}^{+}_{21}$ and calculate $B(E21)$ factors.

Wigner transformation

Wigner transformation

$$f^{\sigma\sigma'}(\mathbf{r},\mathbf{p}) = \int d^3 s e^{-i\mathbf{ps}/\hbar} \left\langle \mathbf{r} + \frac{\mathbf{s}}{2}, \sigma|\hat{\rho}|\mathbf{r} - \frac{\mathbf{s}}{2}, \sigma' \right\rangle .$$

The Wigner transformation of a product of two operators is given by the following formula

$$\begin{aligned} (\hat{h}\hat{\rho})_W &= h(\mathbf{r},\mathbf{p}) \exp\left(\frac{i\hbar}{2} \overleftrightarrow{\Lambda}\right) f(\mathbf{r},\mathbf{p}) \\ &= h(\mathbf{r},\mathbf{p}) f(\mathbf{r},\mathbf{p}) + \frac{i\hbar}{2}\{h,f\} - \frac{\hbar^2}{8}\{\{h,f\}\} + o(\hbar^3), \end{aligned} \quad (18)$$

where $\overleftrightarrow{\Lambda} = \overleftarrow{\nabla}_r \overrightarrow{\nabla}_p - \overleftarrow{\nabla}_p \overrightarrow{\nabla}_r$, $\{h,f\} = h \overleftrightarrow{\Lambda} f$ is the Poisson bracket of functions $h(\mathbf{r},\mathbf{p})$ and $f(\mathbf{r},\mathbf{p})$, $\{\{h,f\}\} = h(\mathbf{r},\mathbf{p})(\overleftrightarrow{\Lambda})^2 f(\mathbf{r},\mathbf{p})$ is their double Poisson bracket.