Giant Dipole and Spin Magnetic Quadrupole Resonances within Wigner Function Moments method

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The Wigner Function Moments (WFM) method is applied to the description of the Giant Dipole Resonance and Related Spin-dependent Excitations in even-even deformed axially symmetric and spherical nuclei.

1. Brief description of the WFM method.
   Details are contained in:

   E. B. Balbutsev, I. V. Molodtsova, and P. Schuck,
   Nucl. Phys. A **872**, 42 (2011), PRC **88**, 014306 (2013),
   PRC **91**, 064312 (2015), PRC **97**, 044316 (2018);

   E. B. Balbutsev, I. V. Molodtsova, A. V. Sushkov, N. Yu. Shirikova, and P. Schuck, PRC **105**, 044323 (2022);

   E. B. Balbutsev and I. V. Molodtsova,
   Eur. Phys. J. A **59**, 207 (2023), Eur. Phys. J. A **60**, 185 (2024).

2. GDR, ESDR and spin-M2 excitations.

3. Concluding remarks.

⇒ The basis of the WFM method is the Time Dependent Hartree-Fock (TDHF) equation for the one-body density matrix $\rho^{\tau}(\mathbf{r},\sigma;\mathbf{r}',\sigma';t) = \langle \mathbf{r},\sigma|\hat{\rho}^{\tau}(t)|\mathbf{r}',\sigma'\rangle$:

$$i\hbar\frac{\partial\hat{\rho}^{\tau}}{\partial t} = \left[\hat{h}^{\tau},\hat{\rho}^{\tau}\right], \tag{1}$$

where $\tau = \{\mathrm{n},\mathrm{p}\}$.

⇒ Microscoic Hamiltonian:

$$H = \sum_{i=1}^{A}\left[\frac{\hat{\mathbf{p}}_i^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}_i^2 - \eta\hat{\mathbf{l}}_i\hat{\mathbf{S}}_i\right] + H_{qq} + H_{dd} + H_{sd},$$

$$H_{qq} = \sum_{\mu=-2}^{2}(-1)^{\mu}\left\{\bar{\kappa}\sum_{i}^{Z}\sum_{j}^{N} + \frac{\kappa}{2}\left[\sum_{i,j(i\neq j)}^{Z} + \sum_{i,j(i\neq j)}^{N}\right]\right\} q_{2-\mu}(\mathbf{r}_i)q_{2\mu}(\mathbf{r}_j),$$

$$H_{dd} = \sum_{\mu=-2}^{2}(-1)^{\mu}\left\{\bar{\xi}\sum_{i}^{Z}\sum_{j}^{N} + \frac{\xi}{2}\left[\sum_{i,j(i\neq j)}^{Z} + \sum_{i,j(i\neq j)}^{N}\right]\right\} r_{-\mu}(\mathbf{r}_i)r_{\mu}(\mathbf{r}_j),$$

$$H_{sd} = \sum_{\lambda=1}^{2}\sum_{\mu}(-1)^{\mu}\left\{\bar{\chi}\sum_{i}^{Z}\sum_{j}^{N} + \frac{\chi}{2}\left[\sum_{i,j(i\neq j)}^{Z} + \sum_{i,j(i\neq j)}^{N}\right]\right\} s_{\lambda-\mu}(i)s_{\lambda\mu}(j),$$

where $q_{2\mu}(\mathbf{r}) = \sqrt{16\pi/5}\, r^2 Y_{2\mu}(\theta,\phi)$, $s_{\lambda\mu} = \{\hat{S}\otimes\hat{r}\}_{\lambda\mu}/\hbar$.

⇒ Fourier (Wigner) transformation

$$f^{\tau,\sigma\sigma'}(\mathbf{r},\mathbf{p},t) = \int d\mathbf{s}\, \mathrm{e}^{-i\mathbf{p}\mathbf{s}/\hbar} \left\langle \mathbf{r} + \frac{\mathbf{s}}{2}, \sigma|\hat{\rho}(t)|\mathbf{r} - \frac{\mathbf{s}}{2}, \sigma' \right\rangle,$$

converts equation (1) for the density matrix into equation for the Wigner function $f^{\tau,\sigma\sigma'}(\mathbf{r},\mathbf{p},t)$, where

$$\sigma\sigma' = \uparrow\uparrow, \downarrow\downarrow, \uparrow\downarrow, \downarrow\uparrow,$$

with the conventional notation $\uparrow$ for $\sigma = \frac{1}{2}$ and $\downarrow$ for $\sigma = -\frac{1}{2}$.

⇒ Integrating the equation for the Wigner function over a phase space with the weights r_μ, p_μ one gets dynamical equations for the collective variables:

$$R_\mu^{\tau,\sigma\sigma'}(t) = (2\pi\hbar)^{-3}\int d\mathbf{r}\int d\mathbf{p}\ r_\mu f^{\tau,\sigma\sigma'}(\mathbf{r},\mathbf{p},t),$$

$$P_\mu^{\tau,\sigma\sigma'}(t) = (2\pi\hbar)^{-3}\int d\mathbf{r}\int d\mathbf{p}\ p_\mu f^{\tau,\sigma\sigma'}(\mathbf{r},\mathbf{p},t). \qquad (2)$$

Only first-rank tensors are taken into account in the present calculations.

⇒ Isoscalar and isovector variables:

$$X_\mu(t) = X^{\rm n}_\mu(t) + X^{\rm p}_\mu(t), \quad \bar{X}_\mu(t) = X^{\rm n}_\mu(t) - X^{\rm p}_\mu(t), \qquad X = \{R, P\}.$$

Spin-scalar and spin-vector variables:

$$X^+_\mu(t) = X^{\uparrow\uparrow}_\mu(t) + X^{\downarrow\downarrow}_\mu(t), \quad X^-_\mu(t) = X^{\uparrow\uparrow}_\mu(t) - X^{\downarrow\downarrow}_\mu(t).$$

⇒ Small amplitude approximation: $X^\varsigma_\mu(t) = X^{\varsigma\,\rm eq}_\mu + \mathcal{X}^\varsigma_\mu(t)$,

$$\mathcal{X}^\varsigma_\mu(t) = \{\mathcal{R}^\varsigma(t), \mathcal{P}^\varsigma(t), \bar{\mathcal{R}}^\varsigma(t), \bar{\mathcal{P}}^\varsigma(t)\}, \quad \varsigma = +, -, \uparrow\downarrow, \downarrow\uparrow.$$

Linearization in deviations $\mathcal{X}_\mu(t)$ and imposing the time evolution via $e^{i\Omega t}$ for all variables allows to transform the system of nonlinear dynamical equations into a set of linear algebraic equations.

Eigenfrequencies Ω are found as solutions of its secular equation.

$K^\pi = 1^-$ equations

$$
\begin{aligned}
\dot{\bar{\mathcal{R}}}_1^+ &= \frac{1}{m}\bar{\mathcal{P}}_1^+ - i\hbar\frac{\eta}{2}\left(\bar{\mathcal{R}}_1^- + \sqrt{2}\bar{\mathcal{R}}_0^{\downarrow\uparrow}\right),\\
\dot{\bar{\mathcal{R}}}_1^- &= \frac{1}{m}\bar{\mathcal{P}}_1^- - i\hbar\frac{\eta}{2}\bar{\mathcal{R}}_1^+,\\
\dot{\bar{\mathcal{R}}}_0^{\downarrow\uparrow} &= \frac{1}{m}\bar{\mathcal{P}}_0^{\downarrow\uparrow} - i\hbar\frac{\eta}{2\sqrt{2}}\bar{\mathcal{R}}_1^+,\\
\dot{\bar{\mathcal{P}}}_1^+ &= -\left(m\omega^2 - 2\kappa_0 Q_{20}\right)\bar{\mathcal{R}}_1^+ - i\hbar\frac{\eta}{2}\left(\bar{\mathcal{P}}_1^- + \sqrt{2}\bar{\mathcal{P}}_0^{\downarrow\uparrow}\right) + 2\alpha\kappa_0 Q_{20} X \mathcal{R}_1^+ - A\left(\xi_1\bar{\mathcal{R}}_1^+ + X\xi_0\mathcal{R}_1^+\right),\\
\dot{\bar{\mathcal{P}}}_1^- &= -\left(m\omega^2 - 2\kappa_0 Q_{20}\right)\bar{\mathcal{R}}_1^- - i\hbar\frac{\eta}{2}\bar{\mathcal{P}}_1^+ + 2\alpha\kappa_0 \bar{Q}_{20}\mathcal{R}_1^- - \frac{A}{4}\left[\chi_1\bar{\mathcal{R}}_1^- + X\chi_0\mathcal{R}_1^-\right],\\
\dot{\bar{\mathcal{P}}}_0^{\downarrow\uparrow} &= -\left(m\omega^2 + 4\kappa_0 Q_{20}\right)\bar{\mathcal{R}}_0^{\downarrow\uparrow} - i\hbar\frac{\eta}{2\sqrt{2}}\bar{\mathcal{P}}_1^+ - 4\alpha\kappa_0 Q_{20} X \mathcal{R}_0^{\downarrow\uparrow} - \frac{A}{4}\left[\chi_1\bar{\mathcal{R}}_0^{\downarrow\uparrow} + X\chi_0\mathcal{R}_0^{\downarrow\uparrow}\right],\\
\dot{\mathcal{R}}_1^+ &= \frac{1}{m}\mathcal{P}_1^+ - i\hbar\frac{\eta}{2}\left(\mathcal{R}_1^- + \sqrt{2}\mathcal{R}_0^{\downarrow\uparrow}\right),\\
\dot{\mathcal{R}}_1^- &= \frac{1}{m}\mathcal{P}_1^- - i\hbar\frac{\eta}{2}\mathcal{R}_1^+,\\
\dot{\mathcal{R}}_0^{\downarrow\uparrow} &= \frac{1}{m}\mathcal{P}_0^{\downarrow\uparrow} - i\hbar\frac{\eta}{2\sqrt{2}}\mathcal{R}_1^+,\\
\dot{\mathcal{P}}_1^+ &= -\left(m\omega^2 - 2\kappa_0 Q_{20}\right)\mathcal{R}_1^+ - i\hbar\frac{\eta}{2}\left(\mathcal{P}_1^- + \sqrt{2}\mathcal{P}_0^{\downarrow\uparrow}\right) + 2\alpha\kappa_0 Q_{20} X \bar{\mathcal{R}}_1^+ - A\left(\xi_0\mathcal{R}_1^+ + X\xi_1\bar{\mathcal{R}}_1^+\right),\\
\dot{\mathcal{P}}_1^- &= -\left(m\omega^2 - 2\kappa_0 Q_{20}\right)\mathcal{R}_1^- - i\hbar\frac{\eta}{2}\mathcal{P}_1^+ + 2\alpha\kappa_0 Q_{20} X \bar{\mathcal{R}}_1^- - \frac{A}{4}\left[\chi_0\mathcal{R}_1^- + X\chi_1\bar{\mathcal{R}}_1^-\right],\\
\dot{\mathcal{P}}_0^{\downarrow\uparrow} &= -\left(m\omega^2 + 4\kappa_0 Q_{20}\right)\mathcal{R}_0^{\downarrow\uparrow} - i\hbar\frac{\eta}{2\sqrt{2}}\mathcal{P}_1^+ - 4\alpha\kappa_0 Q_{20} X \bar{\mathcal{R}}_0^{\downarrow\uparrow} - \frac{A}{4}\left[\chi_0\mathcal{R}_0^{\downarrow\uparrow} + X\chi_1\bar{\mathcal{R}}_0^{\downarrow\uparrow}\right],
\end{aligned}
\qquad (3)
$$

where $X = (N - Z)/A$ is asymmetry parameter, $Q_{20} = \frac{4}{3}\delta Q_{00}$, $Q_{20}^{\tau} = \frac{N^{\tau}}{A}Q_{20}$, $R_0 = 1.2A^{1/3}$,
$\kappa_0 = -m\tilde{\omega}^2/(4Q_{00})$, $\kappa_1 = \alpha\kappa_0$, $\tilde{\omega}^2 = \omega_0^2/[(1 + \frac{4}{3}\delta)^{2/3}(1 - \frac{2}{3}\delta)^{1/3}]$.
$\eta = 2\hbar\omega_0\kappa_{\mathrm{N}}$ and κ_{N} is Nilsson spin-orbital strength constant.

$$|\langle\psi_a|\hat{O}|\psi_0\rangle|^2 = \hbar \lim_{\Omega\to\Omega_a}(\Omega-\Omega_a)\overline{\langle\psi'|\hat{O}|\psi'\rangle \mathrm{e}^{-i\Omega t}}, \qquad \hat{O}(t) = \hat{O}\,\mathrm{e}^{-i\Omega t} + \hat{O}^\dagger\, e^{i\Omega t}$$

A. M. Lane, *Nuclear Theory* (Benjamin, New York, 1964).

$$\hat{O}(E1,\mu) = e\sqrt{\frac{3}{4\pi}}\left(\frac{N}{A}\sum_i^Z - \frac{Z}{A}\sum_i^N\right)[r_\mu]_i$$

$$\hat{O}(M2,\mu) = \mu_N\sqrt{\frac{15}{2\pi}}\sum_i^A\sum_{\nu=-1}^{1} C^{2\mu}_{1(\mu-\nu),1\nu}\left[g_s\hat{S}_\nu r_{\mu-\nu} + \frac{2}{3}g_l\hat{l}_\nu r_{\mu-\nu}\right]_i$$

$$\langle\psi'|\hat{O}(E1,\mu)|\psi'\rangle = e\sqrt{\frac{3}{4\pi}}\left[\frac{N}{A}\mathcal{R}^{\mathrm{p}}_\mu - \frac{Z}{A}\mathcal{R}^{\mathrm{n}}_\mu\right] = -\frac{e}{2}\sqrt{\frac{3}{4\pi}}\left[\bar{\mathcal{R}}^+_\mu - X\mathcal{R}^+_\mu\right]$$

$$\langle\psi'|\hat{O}(M2,\mu)|\psi'\rangle = \mu_N\sqrt{\frac{15}{2\pi}}\sum_\tau\sum_{\nu=-1}^{1} C^{2\mu}_{1(\mu-\nu),1\nu}\left[g_s^\tau\sum_{\sigma,\sigma'}\langle\sigma|\hat{S}_\nu|\sigma'\rangle\mathcal{R}^{\tau,\sigma'\sigma}_{\mu-\nu} - i\frac{2\sqrt{2}}{3}g_l^\tau\mathcal{T}^{\tau,+}_{1\nu,\mu-\nu}\right],$$

$$\text{where}\quad \mathcal{T}^+_{\lambda\mu,\nu}(t) = (2\pi\hbar)^{-3}\int d\mathbf{r}\int d\mathbf{p}\,\{r\otimes p\}_{\lambda\mu}r_\nu\,\delta f^+(\mathbf{r},\mathbf{p},t).$$

$$\langle\psi'|\hat{O}_\sigma(M2,0)|\psi'\rangle = \mu_N\frac{\hbar}{2}\sqrt{\frac{5}{2\pi}}\left[g_s^{\mathrm{is}}\left(\mathcal{R}^-_0 + \frac{\mathcal{R}^{\uparrow\downarrow}_1 - \mathcal{R}^{\downarrow\uparrow}_{-1}}{\sqrt{2}}\right) + g_s^{\mathrm{iv}}\left(\bar{\mathcal{R}}^-_0 + \frac{\bar{\mathcal{R}}^{\uparrow\downarrow}_1 - \bar{\mathcal{R}}^{\downarrow\uparrow}_{-1}}{\sqrt{2}}\right)\right],$$

$$\langle\psi'|\hat{O}_\sigma(M2,1)|\psi'\rangle = \mu_N\frac{\hbar}{2}\sqrt{\frac{15}{8\pi}}\left[g_s^{\mathrm{is}}\left(\mathcal{R}^-_1 - \sqrt{2}\,\mathcal{R}^{\downarrow\uparrow}_0\right) + g_s^{\mathrm{iv}}\left(\bar{\mathcal{R}}^-_1 - \sqrt{2}\,\bar{\mathcal{R}}^{\downarrow\uparrow}_0\right)\right],$$

$$\langle\psi'|\hat{O}_\sigma(M2,2)|\psi'\rangle = -\mu_N\frac{\hbar}{2}\sqrt{\frac{15}{4\pi}}\left[g_s^{\mathrm{is}}\mathcal{R}^{\downarrow\uparrow}_1 + g_s^{\mathrm{iv}}\bar{\mathcal{R}}^{\downarrow\uparrow}_1\right],$$

Giant Dipole Resonance ($\eta = 0$ limit)

$K = 1$

$$\dot{\bar{\mathcal{R}}}_1^+ = \frac{1}{m}\bar{\mathcal{P}}_1^+,$$

$$\dot{\bar{\mathcal{P}}}_1^+ = -m\omega^2\left(1 + \frac{2}{3}\delta\frac{\tilde{\omega}^2}{\omega^2}\right)\bar{\mathcal{R}}_1^+$$
$$-\frac{2}{3}\alpha\delta X m\tilde{\omega}^2\mathcal{R}_1^+ - A\left(\xi_1^{K=1}\bar{\mathcal{R}}_1^+ + X\xi_0^{K=1}\mathcal{R}_1^+\right),$$

$$\dot{\mathcal{R}}_1^+ = \frac{1}{m}\mathcal{P}_1^+,$$

$$\dot{\mathcal{P}}_1^+ = -m\omega^2\left(1 + \frac{2}{3}\delta\frac{\tilde{\omega}^2}{\omega^2}\right)\mathcal{R}_1^+$$
$$-\frac{2}{3}\alpha\delta X m\tilde{\omega}^2\bar{\mathcal{R}}_1^+ - A\left(\xi_0^{K=1}\mathcal{R}_1^+ + X\xi_1^{K=1}\bar{\mathcal{R}}_1^+\right).$$

$K = 0$

$$\dot{\bar{\mathcal{R}}}_0^+ = \frac{1}{m}\bar{\mathcal{P}}_0^+,$$

$$\dot{\bar{\mathcal{P}}}_0^+ = -m\omega^2\left(1 - \frac{4}{3}\delta\frac{\tilde{\omega}^2}{\omega^2}\right)\bar{\mathcal{R}}_0^+$$
$$+\frac{4}{3}\alpha\delta X m\tilde{\omega}^2\mathcal{R}_0^+ - A\left(\xi_1^{K=0}\bar{\mathcal{R}}_0^+ + X\xi_0^{K=0}\mathcal{R}_0^+\right),$$

$$\dot{\mathcal{R}}_0^+ = \frac{1}{m}\mathcal{P}_0^+,$$

$$\dot{\mathcal{P}}_0^+ = -m\omega^2\left(1 - \frac{4}{3}\delta\frac{\tilde{\omega}^2}{\omega^2}\right)\mathcal{R}_0^+$$
$$+\frac{4}{3}\alpha\delta X m\tilde{\omega}^2\bar{\mathcal{R}}_0^+ - A\left(\xi_0^{K=0}\mathcal{R}_0^+ + X\xi_1^{K=0}\bar{\mathcal{R}}_0^+\right).$$

$$X = (N - Z)/A$$

Giant Dipole Resonance ($\eta = 0$ limit)

$$X = 0$$

$$\left[\Omega_{\rm is}^K\right]^2 = \omega^2 C_K + \frac{A}{m}\xi_0^K, \qquad \left[\Omega_{\rm iv}^K\right]^2 = \omega^2 C_K + \frac{A}{m}\xi_1^K$$

$$C_0 = \left(1 - \tfrac{4}{3}\delta\frac{\tilde{\omega}^2}{\omega^2}\right), \quad C_1 = \left(1 + \tfrac{2}{3}\delta\frac{\tilde{\omega}^2}{\omega^2}\right)$$

$$\boxed{\Omega_{\rm is}^K(\tilde{\xi}_0^K) = 0 \;\longrightarrow\; \tilde{\xi}_0^K = -\frac{m\omega^2}{A}C_K}$$

$$\xi_1^K = \xi_{\rm BM} C_K \quad \text{with} \quad \xi_{\rm BM} = 113 A^{-5/3}\ \mathrm{MeV}\cdot\mathrm{fm}^{-2}$$

$$B(E1K)_{\rm iv} = \frac{3}{8\pi}\frac{NZ}{A}\frac{e^2\hbar}{m\Omega_{\rm iv}^K}, \qquad B(E1K)_{\rm is} = 0$$

$$\text{EWSR:} \quad S(E1) = \frac{9}{8\pi}\frac{\hbar^2}{m}\frac{NZ}{A}e^2$$

The numerical estimates are given on the example of the nucleus ^{164}Dy with the deformation parameter $\delta = 0.26$.

$$\tilde{\xi}_0^{K=1} = -0.0126\ \mathrm{MeV}\cdot\mathrm{fm}^{-2}, \qquad \xi_1^{K=1} = 0.0264\ \mathrm{MeV}\cdot\mathrm{fm}^{-2}$$
$$\tilde{\xi}_0^{K=0} = -0.0077\ \mathrm{MeV}\cdot\mathrm{fm}^{-2}, \qquad \xi_1^{K=0} = 0.0162\ \mathrm{MeV}\cdot\mathrm{fm}^{-2}.$$

Giant Dipole Resonance ($\eta = 0$ limit)

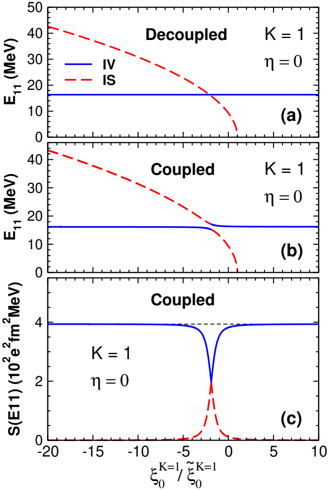


Figure: E_{1K} (a, b, d, e) and $S(E1K) = E_{1K}B(E1K)$ (c, f) vs. $\xi_0^K/\tilde{\xi}_0^K$ ratio. The calculations were performed without spin-orbit interaction. The solid blue lines correspond to GDRs, and the dashed red lines to CMMs. The black short dashed lines in the panels (c) and (f) indicate the EWSR value.

Giant Dipole Resonance ($\eta = 0$ limit)

^{164}Dy

Coupled		$K^\pi = 1^-$	Decoupled ($X = 0$)		
E_1, MeV	$B(E11)$, e^2fm^2		E_1, MeV	$B(E11)$, e^2fm^2	
		$\xi_0^{K=1}/\tilde{\xi}_0^{K=1} = 1$			
0.00	–		0.00	0.00	IS
16.28	23.86		16.35	24.07	IV
		$\xi_0^{K=1}/\tilde{\xi}_0^{K=1} = -30$			
52.56	0.00		51.78	0.00	IS
16.16	24.36		16.35	24.07	IV

Coupled		$K^\pi = 0^-$	Decoupled ($X = 0$)		
E_0, MeV	$B(E10)$, e^2fm^2		E_0, MeV	$B(E10)$, e^2fm^2	
		$\xi_0^{K=0}/\tilde{\xi}_0^{K=0} = 1$			
0.00	–		0.00	0.00	IS
12.80	14.85		12.81	15.36	IV
		$\xi_0^{K=0}/\tilde{\xi}_0^{K=0} = -30$			
40.76	0.00		40.57	0.00	IS
12.56	15.67		12.81	15.36	IV

The experimental GDR energy centroid in medium and heavy nuclei follows a dependence $E_{\rm GDR} = 80A^{-1/3}$ [MeV]. The energy separation caused by deformation was found to be proportional to the ground state deformation δ: $\Delta_{\rm GDR} \simeq E_{\rm GDR}\delta$ [MeV].
For ^{164}Dy, $E_{\rm GDR} = 14.62$ MeV and $\Delta_{\rm GDR} = 3.80$ MeV.

WFM: $\bar{E}_{\rm GDR} = 14.94(14.75)$ MeV and $\Delta_{\rm GDR} = 3.48(3.60)$ MeV.

Spin $M2$ in spherical nuclei

$$\mathcal{R}_0^M = \mathcal{R}_0^- + (\mathcal{R}_1^{\uparrow\downarrow} - \mathcal{R}_{-1}^{\downarrow\uparrow})/\sqrt{2}$$
$$\mathcal{P}_0^M = \mathcal{P}_0^- + (\mathcal{P}_1^{\uparrow\downarrow} - \mathcal{P}_{-1}^{\downarrow\uparrow})/\sqrt{2}$$

$$\begin{aligned}
\dot{\bar{\mathcal{R}}}_0^M &= \frac{1}{m}\bar{\mathcal{P}}_0^M, \\
\dot{\bar{\mathcal{P}}}_0^M &= -m\omega_0^2\bar{\mathcal{R}}_0^M - \frac{A}{4}\left[\chi_1\bar{\mathcal{R}}_0^M + X\chi_0\mathcal{R}_0^M\right], \\
\dot{\mathcal{R}}_0^M &= \frac{1}{m}\mathcal{P}_0^M, \\
\dot{\mathcal{P}}_0^M &= -m\omega_0^2\mathcal{R}_0^M - \frac{A}{4}\left[\chi_0\mathcal{R}_0^M + X\chi_1\bar{\mathcal{R}}_0^M\right],
\end{aligned}$$

$$\Omega_\pm^2 = \omega_0^2 + \frac{A}{8m}\left(\chi_0 + \chi_1 \pm \sqrt{\mathcal{G}}\right),$$

where $\mathcal{G} = (\chi_0 - \chi_1)^2 + 4\chi_0\chi_1 X^2$.

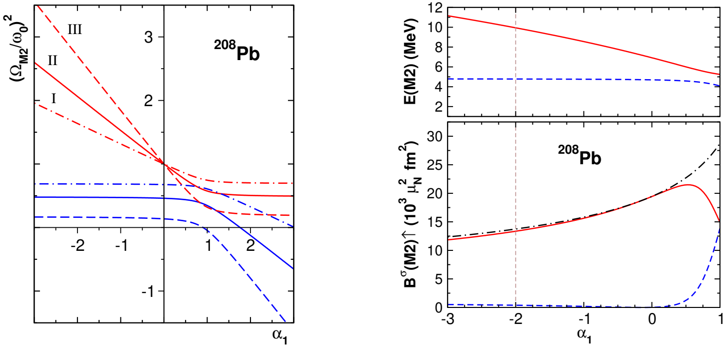
$$B^\sigma(M2)_\pm = 5\frac{15}{32\pi}\frac{\hbar\mu_N^2}{m\Omega_\pm}\Big\{\left[N(g_s^{\rm n})^2 + Z(g_s^{\rm p})^2\right]$$
$$\pm\frac{AX}{\sqrt{\mathcal{G}}}\Big(\left[(g_s^{\rm n})^2 - (g_s^{\rm p})^2\right](\chi_0+\chi_1) + X\left[(g_s^{\rm n}+g_s^{\rm p})^2\chi_1 + (g_s^{\rm n}-g_s^{\rm p})^2\chi_0\right]\Big)\Big\}$$

$$S_1^\sigma(M2) = 5\frac{15}{16\pi}\frac{\hbar^2}{m}\left[(g_s^{\rm n})^2 N + (g_s^{\rm p})^2 Z\right]\mu_N^2$$

$$g_s^{\rm p(n)} = 5.586(-3.826) \longrightarrow g_s^{\rm iv(is)} = 9.412(1.760)$$

$$B^\sigma(M2) \sim g_s^2 \qquad \left[g_s = 0.7 g_s^{\rm free}\right]$$

The spin-dipole–spin-dipole coupling constants: $\chi_0 = \frac{4\pi\kappa_{\rm sd}}{A\langle r^2\rangle}$ MeV/fm^2 and $\chi_1 = \alpha_1\chi_0$.
Castel and Hamamoto, Phys. Lett. B **65**, 27 (1976): $\kappa_{\rm sd} = 25$ MeV.

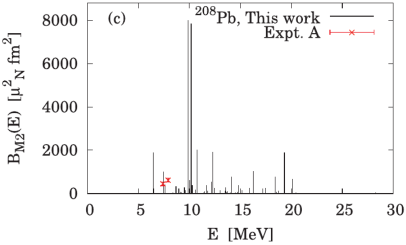


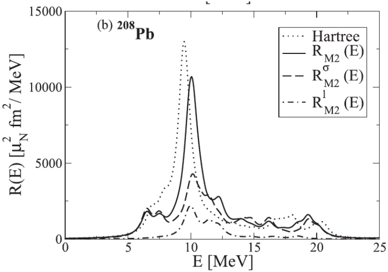
The $(\Omega_{M2}/\omega_0)^2$ vs. $\alpha_1 = \chi_1/\chi_0$. The red/blue lines mark $\Omega_\pm$ for different $\kappa_{\rm sd}$ values: I – 15 MeV, II – 25 MeV, III – 40 MeV.

Energies $E(M2)$ and $B^\sigma(M2)$ strengths as a function of α_1. $E_1 = \hbar\Omega_+$ (red line), $E_2 = \hbar\Omega_-$ (blue dashed line). The dash-dot line shows the summed $M2$ strength.

	WFM		EPJA **59**, 50 (2023)		
Nuclei	$E(2^-)$, MeV	$B^\sigma(M2)$, μ_N^2fm^2	$E(2^-)$, MeV	$B(M2)$, μ_N^2fm^2	$\sum B^\sigma(M2)$
^{208}Pb	4.78	386.50	6.46	1815	25515
	9.94	13353.42	$\sim$ 10	15863.82	
^{90}Zr	6.24	313.82	$\sim$ 5	11780	
	13.18	6464.66	$\sim$ 13		

The properties of $M2$ transitions were studied in the framework of RQRPA in Ref. [Kružić *et. al*, Eur. Phys. J. A **59**, 50 (2023)].





[Kružić *et. al*, EPJA **59**, 50 (2023)].
Expt.: [Lindgren *et. al*, PRL **36**, 116 (1976)].

Electric $K^\pi = 0^-$

$\mathcal{R}_0^+, \mathcal{P}_0^+, \quad \mathcal{R}_0^E = \mathcal{R}_1^{\uparrow\downarrow} + \mathcal{R}_{-1}^{\downarrow\uparrow}, \mathcal{P}_0^E = \mathcal{P}_1^{\uparrow\downarrow} + \mathcal{P}_{-1}^{\downarrow\uparrow}$

$$\dot{\bar{\mathcal{R}}}_0^+ = \frac{1}{m}\bar{\mathcal{P}}_0^+ - i\hbar\frac{\eta}{2}\sqrt{2}\bar{\mathcal{R}}_0^E,$$

$$\dot{\bar{\mathcal{R}}}_0^E = \frac{1}{m}\bar{\mathcal{P}}_0^E - i\hbar\frac{\eta}{2}\sqrt{2}\,\bar{\mathcal{R}}_0^+,$$

$$\dot{\bar{\mathcal{P}}}_0^+ = -m\omega^2 C_0\bar{\mathcal{R}}_0^+ - i\hbar\frac{\eta}{2}\sqrt{2}\bar{\mathcal{P}}_0^E + \frac{4}{3}\delta X m\tilde{\omega}^2\mathcal{R}_0^+ - A\left(\xi_1\bar{\mathcal{R}}_0^+ + X\xi_0\mathcal{R}_0^+\right),$$

$$\dot{\bar{\mathcal{P}}}_0^E = -m\omega^2 C_1\bar{\mathcal{R}}_0^E - i\hbar\frac{\eta}{2}\sqrt{2}\,\bar{\mathcal{P}}_0^+ - \frac{2}{3}\delta X m\tilde{\omega}^2\mathcal{R}_0^E - \frac{A}{4}\left[\chi_1\bar{\mathcal{R}}_0^E + X\chi_0\mathcal{R}_0^E\right],$$

$$\dot{\mathcal{R}}_0^+ = \frac{1}{m}\mathcal{P}_0^+ - i\hbar\frac{\eta}{2}\sqrt{2}\mathcal{R}_0^E,$$

$$\dot{\mathcal{R}}_0^E = \frac{1}{m}\mathcal{P}_0^E - i\hbar\frac{\eta}{2}\sqrt{2}\,\mathcal{R}_0^+,$$

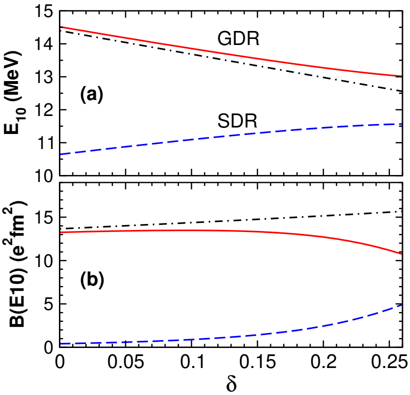
$$\dot{\mathcal{P}}_0^+ = -m\omega^2 C_0\mathcal{R}_0^+ - i\hbar\frac{\eta}{2}\sqrt{2}\mathcal{P}_0^E + \frac{4}{3}\delta X m\tilde{\omega}^2\bar{\mathcal{R}}_0^+ - A\left(\xi_0\mathcal{R}_0^+ + X\xi_1\bar{\mathcal{R}}_0^+\right),$$

$$\dot{\mathcal{P}}_0^E = -m\omega^2 C_1\mathcal{R}_0^E - i\hbar\frac{\eta}{2}\sqrt{2}\,\mathcal{P}_0^+ - \frac{2}{3}\delta X m\tilde{\omega}^2\bar{\mathcal{R}}_0^E - \frac{A}{4}\left[\chi_0\mathcal{R}_0^E + X\chi_1\bar{\mathcal{R}}_0^E\right].$$

$$\langle\hat{p}_x\hat{\sigma}_y - \hat{p}_y\hat{\sigma}_x\rangle = \mathrm{Tr}\left(\hat{\rho}[\hat{\mathbf{p}}\times\hat{\sigma}]_0\right) = -i\sqrt{2}\left(P_1^{\uparrow\downarrow} + P_{-1}^{\downarrow\uparrow}\right) = -i\sqrt{2}P_0^E,$$

$$\langle x\hat{\sigma}_y - y\hat{\sigma}_x\rangle = \mathrm{Tr}\left(\hat{\rho}[\mathbf{r}\times\hat{\sigma}]_0\right) = -i\sqrt{2}\left(R_1^{\uparrow\downarrow} + R_{-1}^{\downarrow\uparrow}\right) = -i\sqrt{2}R_0^E,$$

where σ_i are Pauli matrices.



WFM calculations for ^{164}Dy:

E_{10}, MeV	$B(E10)$, e^2fm^2
7.57	0.00
11.57	**4.93**
13.01	**10.73**

Figure: (a) Energies of $K = 0$ GDR (red line) and SDR (blue dashed line) with (b) $E1$ strength of GDR (red line) and SDR (blue dashed line) vs. deformation δ. The energy centroid $\bar{E}_{10}$ and summed $B(E10)$ value are shown by black dot-dashed lines.

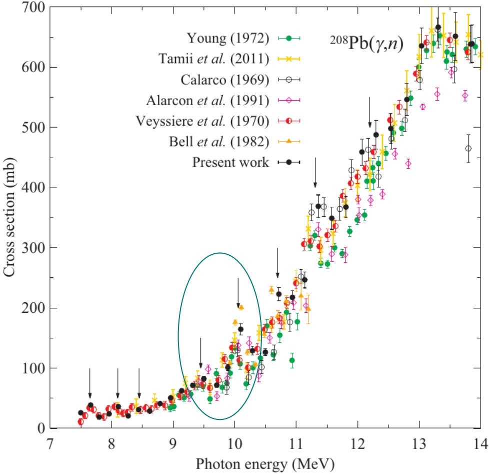
^{208}**Pb**

WFM

$E(1^-)$, MeV	$B(E1)$, e^2fm^2
4.73	0.01
9.83	**1.70**
13.37	**54.32**

✓ The ^{208}Pb(p, p') [Morsch, *et al.*, Nucl. Phys. A **297**, 317 (1978)]: "The strong states at 6.26 and 8.37 MeV yield evidence for a collective 1^- spin-flip excitation."

✓✓ The ^{208}Pb$(p, p'\gamma)$ reaction has been studied in [Wasilewska *et al.*, Phys. Rev. C **105**, 014310 (2022)]. Several discrete states of electrical nature have been detected between 7 and 10 MeV. It was supposed that some of them are collective 1^- spin-flip excitations.



✓✓✓ Experimental ^{208}Pb$(\gamma,\ n)$ cross section in the GDR region [Georghe *et al.*, Phys. Rev. C **110**, 014619 (2024)]. Arrows indicate resonance structures.

Conclusion

The isovector Giant Dipole Resonance and related spin-dependent excitations are studied within the Wigner Function Moments method.

- The energy position and caused by the deformation splitting of GDR are calculated for ^{164}Dy nuclei. Ways to eliminate the excitation of the center of mass motion are discussed.
- The spin degrees of freedom give rise to the electric spin dipole resonance (SDR).
  ✓ The isovector SDR is expected to be seen in the energy region between 9 and 12 MeV, and its gap with the GDR depends on the deformation of the nuclei.
- Spin $M2$ states are studied in ^{208}Pb and ^{90}Zr.
  ✓ The $M2$ transition strength is dominated by the isovector response.
  ✓ The position of $M2$ states is determined by the parameters of spin-dipole–spin-dipole interaction.

THANK YOU

$$\langle \hat{g} \rangle = Tr(\hat{g}\hat{\rho}) = \int d\mathbf{r} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} g(\mathbf{r},\mathbf{p}) f(\mathbf{r},\mathbf{p})$$

$$\langle \hat{g}\hat{\sigma}_i \rangle = Tr(\hat{g}\hat{\sigma}_i\hat{\rho}) = \sum_{s,s'} \int d\mathbf{r} \int d\mathbf{r}' \langle \mathbf{r}|\hat{g}(\mathbf{r})|\mathbf{r}'\rangle \langle s|\hat{\sigma}_i|s'\rangle \langle \mathbf{r}', s'|\hat{\rho}|\mathbf{r}, s\rangle =$$

$$\sum_{s,s'} \langle s|\hat{\sigma}_i|s'\rangle \int d\mathbf{r} \int d\mathbf{r}' \hat{g}(\mathbf{r}) \delta(\mathbf{r}-\mathbf{r}') \langle \mathbf{r}', s'|\hat{\rho}|\mathbf{r}, s\rangle = \sum_{s,s'} \langle s|\hat{\sigma}_i|s'\rangle \int d\mathbf{r} \hat{g}(\mathbf{r}) \langle \mathbf{r}, s'|\hat{\rho}|\mathbf{r}, s\rangle =$$

$$\sum_{s,s'} \langle s|\hat{\sigma}_i|s'\rangle \int d\mathbf{r} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} g(\mathbf{r},\mathbf{p}) f^{s's}(\mathbf{r},\mathbf{p}).$$

$$\langle \hat{p}_{-1}\hat{\sigma}_1 \rangle = -\sqrt{2} P_{-1}^{\downarrow\uparrow}, \qquad \langle \hat{p}_1\hat{\sigma}_{-1} \rangle = \sqrt{2} P_1^{\uparrow\downarrow}, \qquad \langle \hat{p}_0\hat{\sigma}_0 \rangle = P_0^{-}.$$

$$\langle \hat{p}_x\hat{\sigma}_y - \hat{p}_y\hat{\sigma}_x \rangle = \langle [\hat{\mathbf{p}} \times \sigma]_z \rangle = \langle [\hat{\mathbf{p}} \times \sigma]_0 \rangle = i\langle \hat{p}_{-1}\hat{\sigma}_1 - \hat{p}_1\hat{\sigma}_{-1} \rangle = -i\sqrt{2}\left(P_1^{\uparrow\downarrow} - P_{-1}^{\downarrow\uparrow}\right).$$

The TDHF equation reads

$$i\hbar\dot{\hat{\rho}} = \hat{h}\hat{\rho} - \hat{\rho}\hat{h} \tag{4}$$

Let us consider its matrix form in coordinate space keeping all spin indices s, s': $\langle \mathbf{r}, s|\hat{\rho}|\mathbf{r}', s'\rangle$, etc. We do not specify the isospin indices in order to make formulae more transparent. After introduction of the more compact notation $\langle \mathbf{r}, s|\hat{X}|\mathbf{r}', s'\rangle = X_{rr'}^{ss'}$ the set of equations (4) with specified spin indices reads

$$\begin{aligned}
i\hbar\dot{\rho}_{rr''}^{\uparrow\uparrow} &= \int d\mathbf{r}' \left(h_{rr'}^{\uparrow\uparrow}\rho_{r'r''}^{\uparrow\uparrow} - \rho_{rr'}^{\uparrow\uparrow}h_{r'r''}^{\uparrow\uparrow} + h_{rr'}^{\uparrow\downarrow}\rho_{r'r''}^{\downarrow\uparrow} - \rho_{rr'}^{\uparrow\downarrow}h_{r'r''}^{\downarrow\uparrow} \right), \\
i\hbar\dot{\rho}_{rr''}^{\uparrow\downarrow} &= \int d\mathbf{r}' \left(h_{rr'}^{\uparrow\uparrow}\rho_{r'r''}^{\uparrow\downarrow} - \rho_{rr'}^{\uparrow\uparrow}h_{r'r''}^{\uparrow\downarrow} + h_{rr'}^{\uparrow\downarrow}\rho_{r'r''}^{\downarrow\downarrow} - \rho_{rr'}^{\uparrow\downarrow}h_{r'r''}^{\downarrow\downarrow} \right), \\
i\hbar\dot{\rho}_{rr''}^{\downarrow\uparrow} &= \int d\mathbf{r}' \left(h_{rr'}^{\downarrow\uparrow}\rho_{r'r''}^{\uparrow\uparrow} - \rho_{rr'}^{\downarrow\uparrow}h_{r'r''}^{\uparrow\uparrow} + h_{rr'}^{\downarrow\downarrow}\rho_{r'r''}^{\downarrow\uparrow} - \rho_{rr'}^{\downarrow\downarrow}h_{r'r''}^{\downarrow\uparrow} \right), \\
i\hbar\dot{\rho}_{rr''}^{\downarrow\downarrow} &= \int d\mathbf{r}' \left(h_{rr'}^{\downarrow\uparrow}\rho_{r'r''}^{\uparrow\downarrow} - \rho_{rr'}^{\downarrow\uparrow}h_{r'r''}^{\uparrow\downarrow} + h_{rr'}^{\downarrow\downarrow}\rho_{r'r''}^{\downarrow\downarrow} - \rho_{rr'}^{\downarrow\downarrow}h_{r'r''}^{\downarrow\downarrow} \right).
\end{aligned} \tag{5}$$

$$
\begin{aligned}
i\hbar\dot{f}^{\uparrow\uparrow} &= i\hbar\{h^{\uparrow\uparrow}, f^{\uparrow\uparrow}\} + h^{\uparrow\downarrow}f^{\downarrow\uparrow} - f^{\uparrow\downarrow}h^{\downarrow\uparrow} + \frac{i\hbar}{2}\{h^{\uparrow\downarrow}, f^{\downarrow\uparrow}\} - \frac{i\hbar}{2}\{f^{\uparrow\downarrow}, h^{\downarrow\uparrow}\} \\
&- \frac{\hbar^2}{8}\{\{h^{\uparrow\downarrow}, f^{\downarrow\uparrow}\}\} + \frac{\hbar^2}{8}\{\{f^{\uparrow\downarrow}, h^{\downarrow\uparrow}\}\} + \ldots, \\
i\hbar\dot{f}^{\downarrow\downarrow} &= i\hbar\{h^{\downarrow\downarrow}, f^{\downarrow\downarrow}\} + h^{\downarrow\uparrow}f^{\uparrow\downarrow} - f^{\downarrow\uparrow}h^{\uparrow\downarrow} + \frac{i\hbar}{2}\{h^{\downarrow\uparrow}, f^{\uparrow\downarrow}\} - \frac{i\hbar}{2}\{f^{\downarrow\uparrow}, h^{\uparrow\downarrow}\} \\
&- \frac{\hbar^2}{8}\{\{h^{\downarrow\uparrow}, f^{\uparrow\downarrow}\}\} + \frac{\hbar^2}{8}\{\{f^{\downarrow\uparrow}, h^{\uparrow\downarrow}\}\} + \ldots, \\
i\hbar\dot{f}^{\uparrow\downarrow} &= f^{\uparrow\downarrow}h^{-} + \frac{i\hbar}{2}\{h^{+}, f^{\uparrow\downarrow}\} - \frac{\hbar^2}{8}\{\{h^{-}, f^{\uparrow\downarrow}\}\} \\
&- h^{\uparrow\downarrow}f^{-} + \frac{i\hbar}{2}\{h^{\uparrow\downarrow}, f^{+}\} + \frac{\hbar^2}{8}\{\{h^{\uparrow\downarrow}, f^{-}\}\} + \ldots., \\
i\hbar\dot{f}^{\downarrow\uparrow} &= -f^{\downarrow\uparrow}h^{-} + \frac{i\hbar}{2}\{h^{+}, f^{\downarrow\uparrow}\} + \frac{\hbar^2}{8}\{\{h^{-}, f^{\downarrow\uparrow}\}\} \\
&+ h^{\downarrow\uparrow}f^{-} + \frac{i\hbar}{2}\{h^{\downarrow\uparrow}, f^{+}\} - \frac{\hbar^2}{8}\{\{h^{\downarrow\uparrow}, f^{-}\}\} + \ldots, \qquad (6)
\end{aligned}
$$

where the functions $h^{s,s'}(\mathbf{r},\mathbf{p})$, $f^{s,s'}(\mathbf{r},\mathbf{p})$ are the Wigner transforms of $h^{s,s'}_{r,r'}$, $\rho^{s,s'}_{r,r'}$, respectively, $\{f,g\}$ is the Poisson bracket of the functions $f(\mathbf{r},\mathbf{p})$ and $g(\mathbf{r},\mathbf{p})$, $\{\{f,g\}\}$ is their double Poisson bracket, $f^{\pm} = f^{\uparrow\uparrow} \pm f^{\downarrow\downarrow}$ and $h^{\pm} = h^{\uparrow\uparrow} \pm h^{\downarrow\downarrow}$. The dots stand for terms proportional to higher powers of $\hbar$ – after integration over phase space these terms disappear and we arrive to the set of exact integral equations.

Third rank tensors

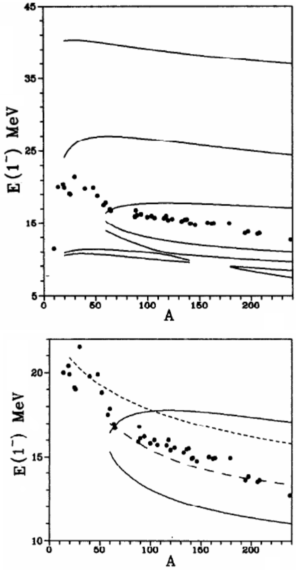
$$\langle\psi'|\hat{O}(M2,\mu)|\psi'\rangle = \mu_N\sqrt{\frac{15}{2\pi}}\sum_{\tau}\sum_{\nu=-1}^{1} C^{2\mu}_{1(\mu-\nu),1\nu}$$

$$\left[g_s^{\tau}\sum_{\sigma,\sigma'}\langle\sigma|\hat{S}_{\nu}|\sigma'\rangle\mathcal{R}^{\tau,\sigma'\sigma}_{\mu-\nu} - i\frac{2\sqrt{2}}{3}g_l^{\tau}\,\mathcal{T}^{\tau,+}_{1\nu,\mu-\nu}\right]$$

orbital twist

$$\mathcal{T}^{\tau,+}_{1\mu,\nu}(t) = \int d\mathbf{r}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\,\{r\otimes p\}_{1\mu}r_{\nu}\,\delta f^{\tau,+}(\mathbf{r},\mathbf{p},t)$$

$$\hat{O}_{ISGQR}(\mu) = e\sum_i\left(r_i^3 - \frac{5}{3}\langle r^2\rangle r_i\right)Y_{1\mu}(\hat{\mathbf{r}}_i)$$



E. B. Balbutsev, J. Piperova, M. Durand, I. V. Molodtsova, and A. V. Unzhakova, Nucl. Phys. A **571**, 413 (1994).