# Neutron star inner crust: advanced liquid drop model

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LXXV International conference Nucleus-2025

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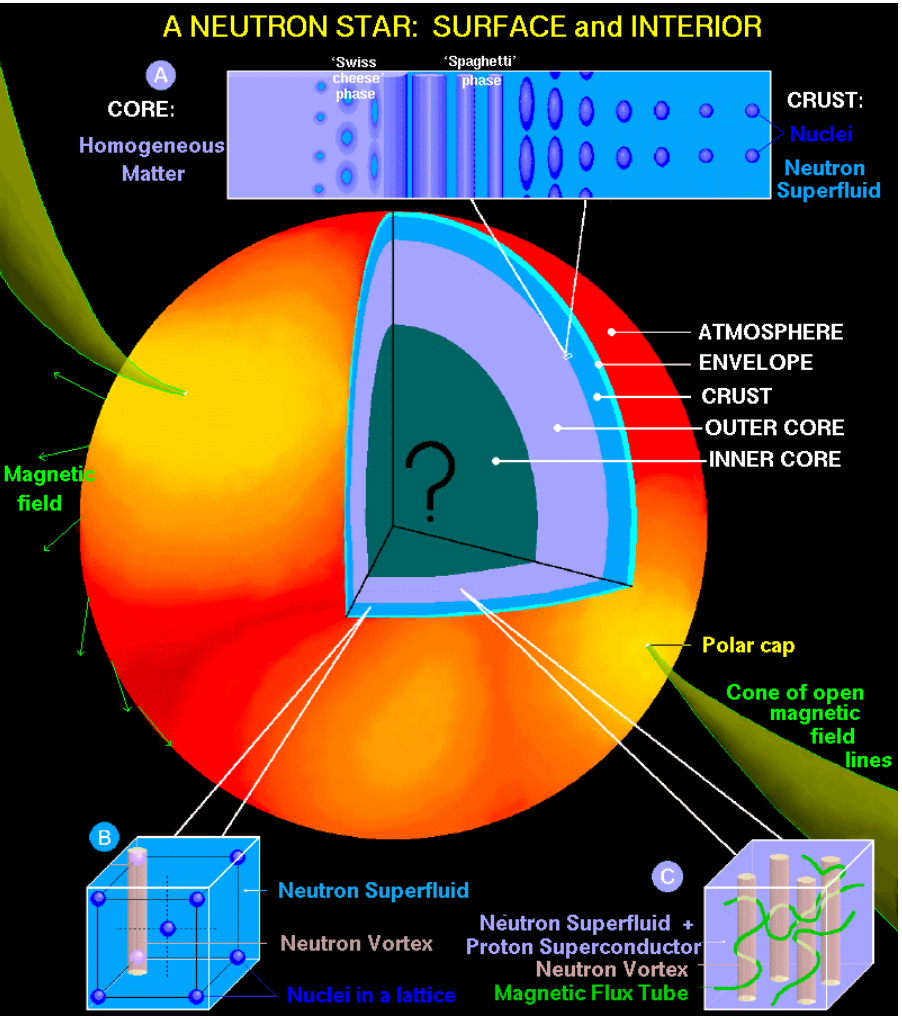
# Neutron star structure

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$$\rho \sim 10^{15}\ \mathrm{g/cm^3}$$

$$T \lesssim 10^{9}\ \mathrm{K}$$

$$B \sim 10^{12}\ \mathrm{G}$$

$$g \sim 10^{14}\ \mathrm{cm/s^2}$$

$$R \sim 2R_{\mathrm{g}} = 4GM/c^2$$

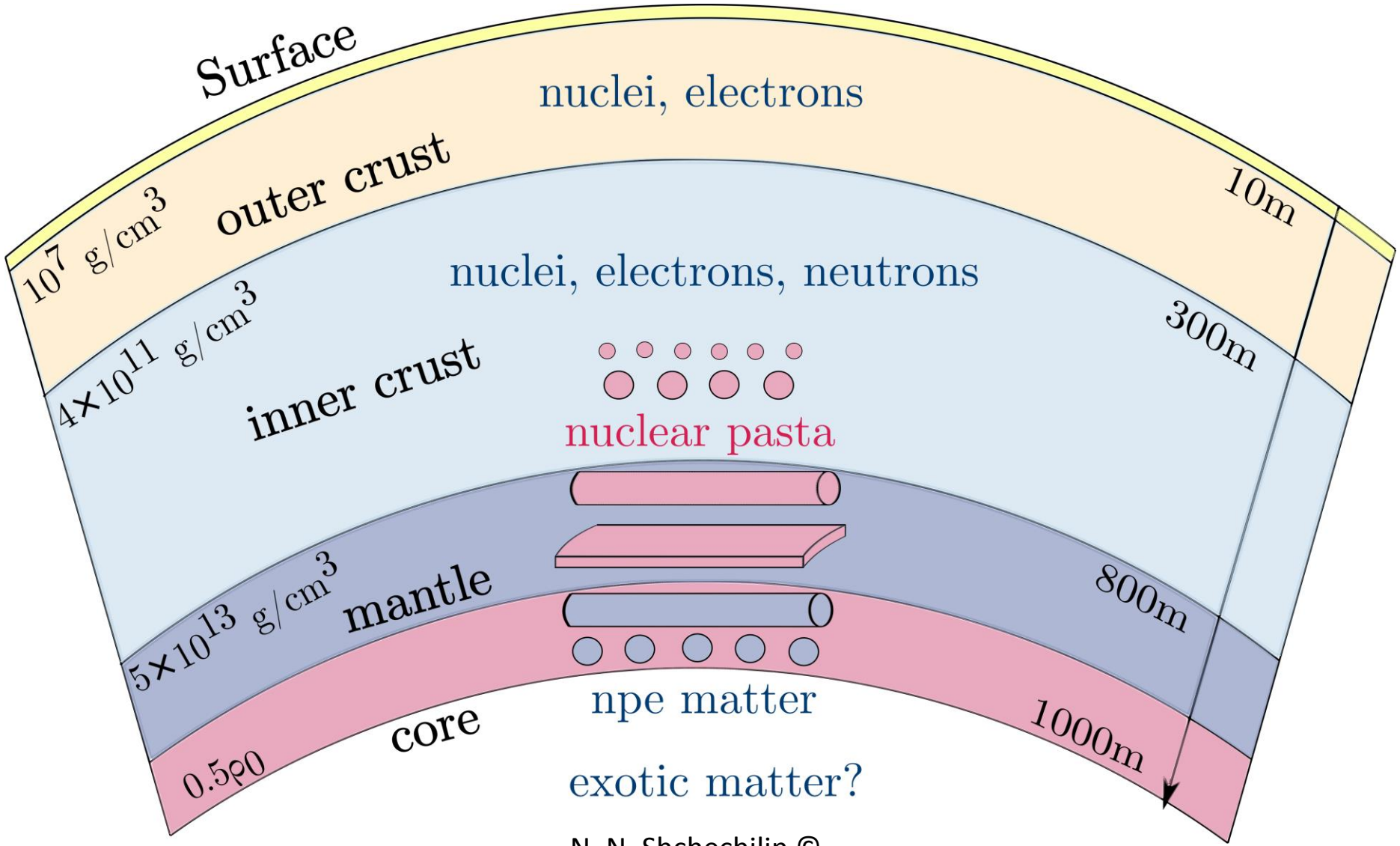
$$T_{\mathrm{cp}} \sim 10^{9}\ \mathrm{K}$$

$$T_{\mathrm{cn}} \sim 10^{8}\ \mathrm{K}$$

$$R \sim 10 - 14\ \mathrm{km},\ M \sim 1.4 M_{\odot}$$

- Neutron star are extreme objects
- They are observed
- Observations are affected by crust

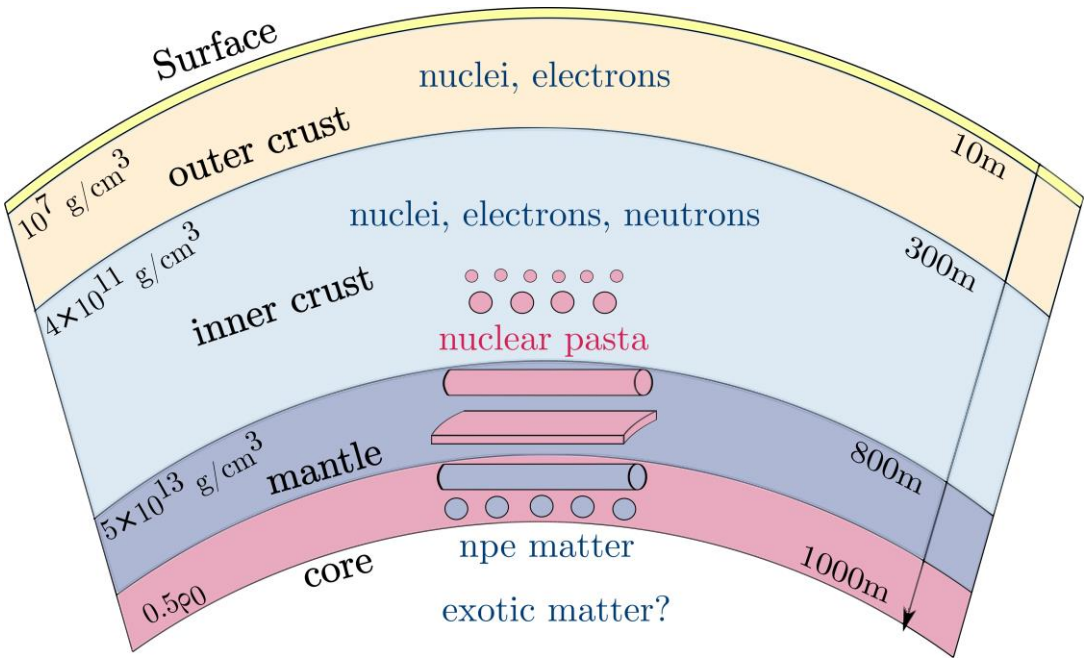
# Neutron star crust




N. N. Shchechilin ©

Crust: nonunifrom nuclear matter **with neutralizing background of electrons**

# That we want to know about the crust?




N. N. Shchechilin ©

- **Composition**
  - Equilibrium
  - Nonequilibrium
- **Equation of state**
  - T=0
  - Thermal properties
  - State of matter (solid/liquid)
- **Dynamical properties One/two liquid hydro (magneto) dynamics**
- **Transport properties (kinetic coefficients )**
- **Elasticity, strength**
- ...

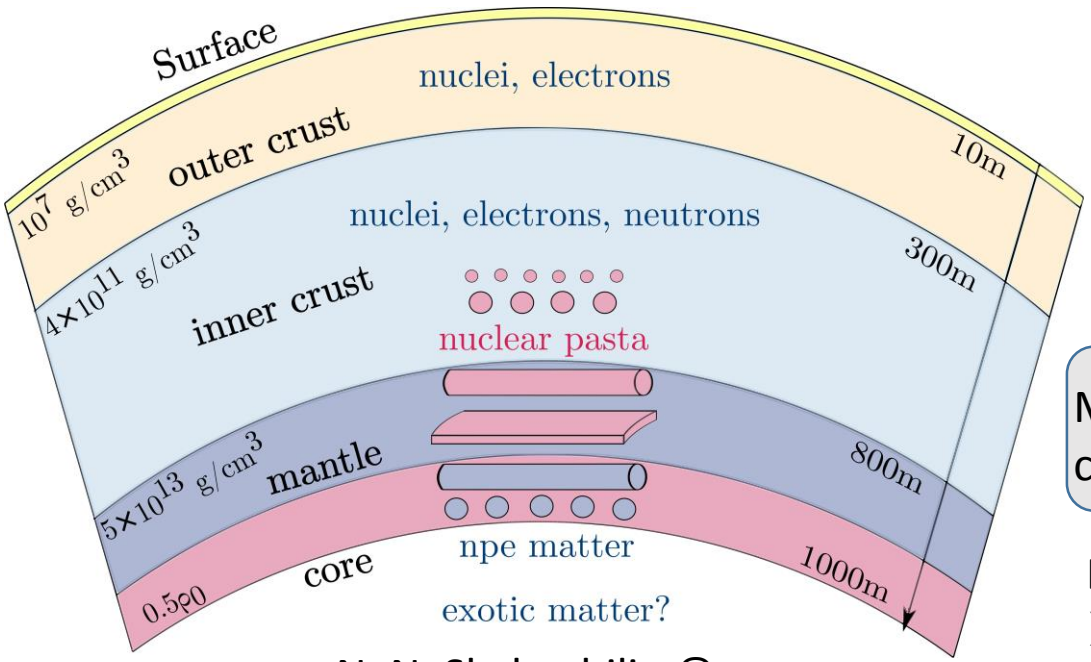
D.G. Yakovlev, HEA2017(?)
Crust as Cinderella of NS

## Why???

These properties affect observations, and thus they are required for adequate interpretation of observations

Typically: the main mystery of NSs is the core. The crustal properties should be known accurately to avoid biases for the core properties

# That we want to know about the crust?




N. N. Shchechilin ©

- **Composition**
  - Equilibrium
  - Nonequilibrium
- **Equation of state**
  - T=0

Matter is neutral: proton charge is compensated by degenerate electrons

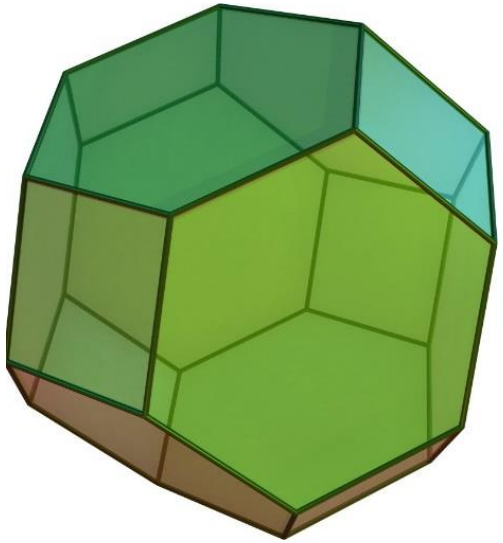
Energy density as function of:
- Nucleon number density
- Nuclei number density
- Nuclei charge Z

<u>Assignment:</u> calculate these properties and provide a tractable procedure to include results into applications. *Results should be trustworthy*!

<u>Approach:</u>
- Elaborate the compressible liquid drop model to make it enough accurate for applications

# Wigner-Seitz (WS) approximation: Spheres instead as unit cells

Clusters form a perfect regular lattice



Energy per cluster = energy of the cell

WS approximation: consider spherical cells

Simplify the analysis: the boundary conditions, symmetry

# Concrete formulation of problem

Provide an approach to calculate for astrophysical applications (for example, to be incorporated into reaction networks)

$$\epsilon(n_b, n_N, Z)$$

"Gold standard": Extended Thomas-Fermi approach

Energy density functional (Wigner-Kirkwood expansion of the Bloch density matrix)

Kirzhnits (1957), Hodges (1973), Grammaticos & Voros (1979), Brack et al. (1985)

$$E_{\mathrm{ETF}} = \int_{\mathrm{cell}} (\tau_4 + \epsilon_{\mathrm{nuc}} + \epsilon_C + \epsilon_e)\, dV$$

kinetic nuclear Coulomb electron

Nucleon profile optimization

Problem:

*Too high computational costs to be incorporated into the codes*

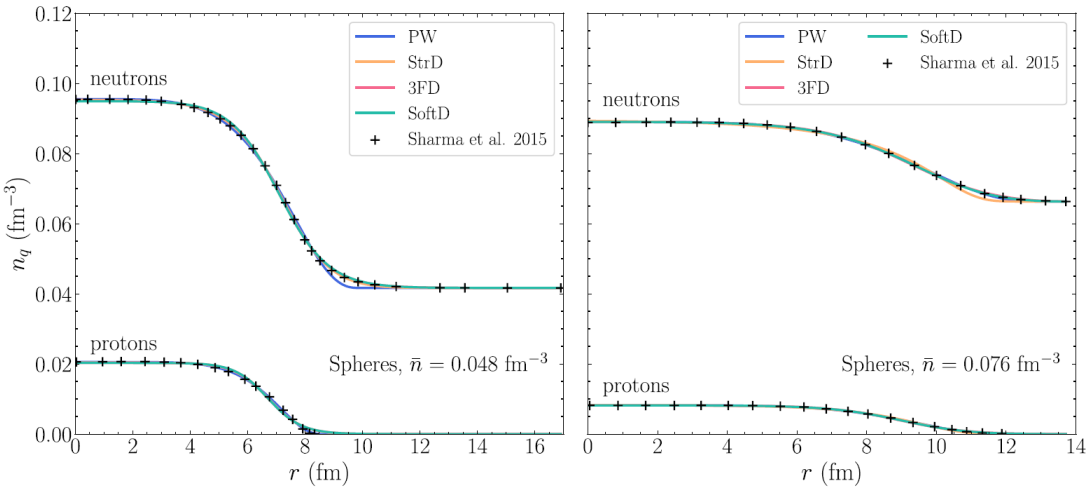


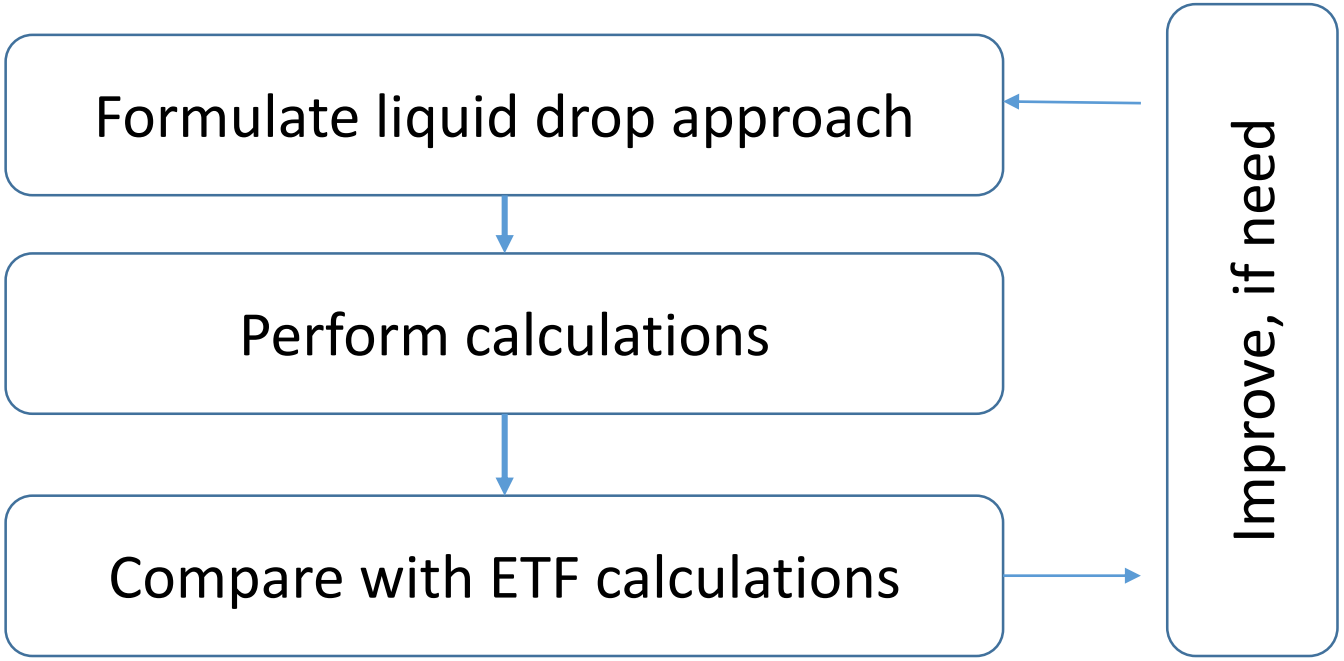

Figure from Shchechilin et al. (2024)

# Concrete formulation of problem

Provide an approach to calculate for astrophysical applications (for example, to be incorporated into reaction networks)

$$\epsilon(n_b, n_N, Z)$$

**Aim of this talk: a simplified description within liquid drop model**



# Compressible liquid drop model for inner crust

You're probably thinking: the liquid drop model has been known for ages!!!
Now It is the 21 century and we are at LXXV International conference Nucleus-2025!!!

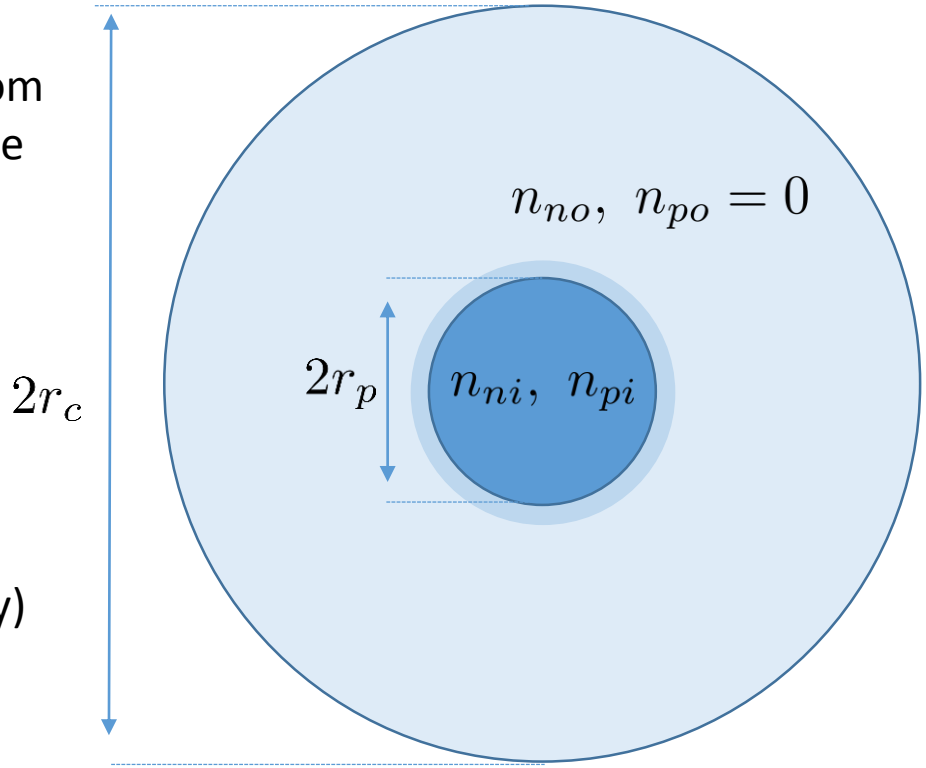
Short answer: You are perfectly right! We are elaborating details to improve precision for nowdays needs and are happy to present the results on the LXXV International conference Nucleus-2025 (we have achieved precision of a few keV/nucleon)

# Compressible liquid drop model for inner crust

- CLDM is rather a class of modes, than one model
- CLDM does not assume that step-like profile for proton and neutron density is real. Rather, general feature of CLDMs is that they start from **explicit analytical expression for energy** of the cell, written as simple as possible, but (reasonable) accurate

$$E = E_{\rm in} + E_{\rm out} + E_{\rm surf} + E_C + E_e$$

- Explicit expressions for all thermodynamic quantities can be obtained analytically. It guaranties absolute (up to numerical accuracy) thermodynamic consistency of the model (if properly applied)




"Everything should be made as simple as possible, but not simpler"

Attributed to Albert Einstein

# Compressible liquid drop model for inner crust

- CLDM is rather a class of modes, than one model
- General feature of these models is that they start from explicit analytical expression (for energy)

$$E = E_{\rm in} + E_{\rm out} + E_{\rm surf} + E_C + E_e$$

Nuclear energy inside cluster:

$$E_{\rm in} = \epsilon_{\rm nm}(n_{ni}, n_{pi}) V_i$$

Nuclear energy outside cluster:

$$E_{\rm out} = \epsilon_{\rm nm}(n_{no}, n_{po}) V_o$$

Surface energy

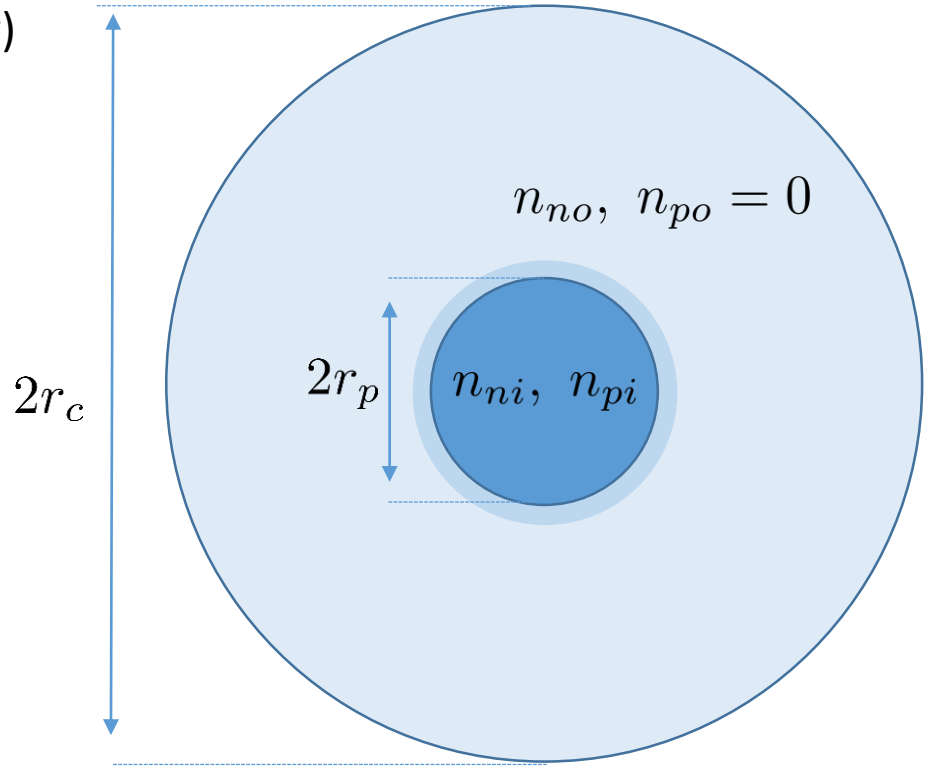
$$E_{\rm surf} = \sigma(\ldots) S + \ldots$$

Coulomb energy: two uniformly charged balls

$$E_C = \frac{3}{5}\frac{Z^2 e^2}{R_N}\left(1 - \frac{3}{2}u^{1/3} + \frac{1}{2}u\right) + \ldots$$

Energy of electrons (uniform degenerate ideal gas)

$$E_e = \epsilon_e(n_e)(V_o + V_i) + E_e^{\rm ex}$$



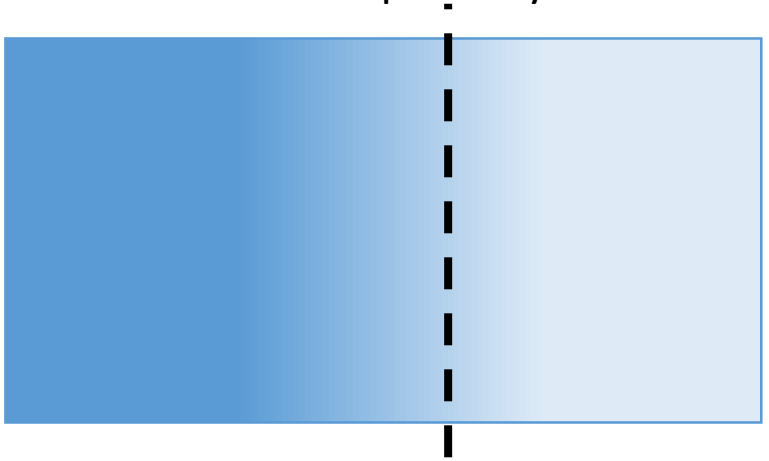

Surface energy? What is it?
Something artificial???

# Surface energy is natural. It can be calculated

Centelles et al. Nucl. Phys. A, 635 (1998), 193

Realistic two-phase system

Reference two-phase system



$E_{\text{two phase}}, N_{\text{two phase}}, Z_{\text{two phase}}$

# Surface energy is natural. It can be calculated

Centelles et al. Nucl. Phys. A, 635 (1998), 193

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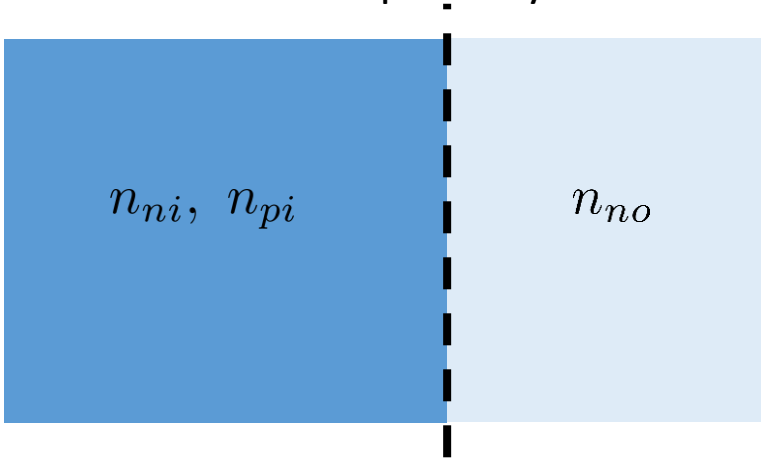



$$E_{\text{two phase}}, N_{\text{two phase}}, Z_{\text{two phase}}$$

$$\begin{aligned} E_{\text{ref}} &= \epsilon_i V_i + \epsilon_o V_o, \\ N_{\text{ref}} &= n_{ni} V_i + n_{no} V_o, \\ Z_{\text{ref}} &= n_{pi} V_i \end{aligned}$$

$$\begin{aligned} E_{\text{surf}} &= E_{\text{two phase}} - E_{\text{ref}} \\ N_{\text{surf}} &= N_{\text{two phase}} - N_{\text{ref}} \\ Z_{\text{surf}} &= Z_{\text{two phase}} - Z_{\text{ref}} \end{aligned}$$

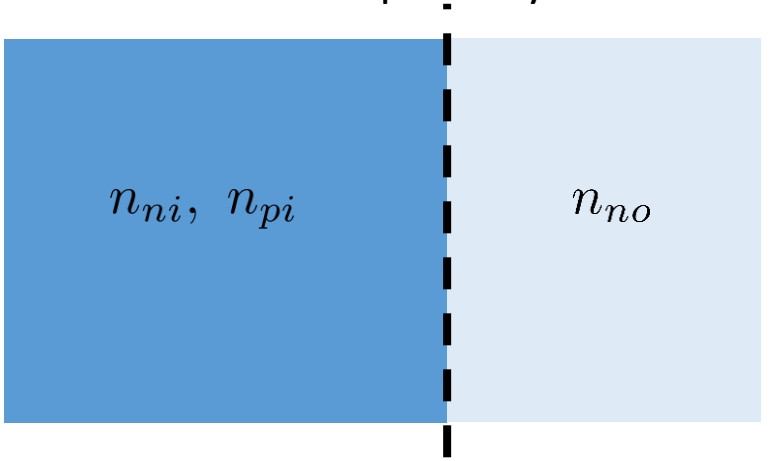
Can be nulled by choose of the reference system

- Surface energy is a correction, required to reproduce two phase system energy
- (Neutron) adsorption is required for thermodynamically consistent description of two phase boundary
- **Surface energy describe two-phase thermodynamics precisely**

# Surface energy is natural. It can be calculated

Realistic two-phase system

Reference two-phase system

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Can be nulled by choose of the reference system

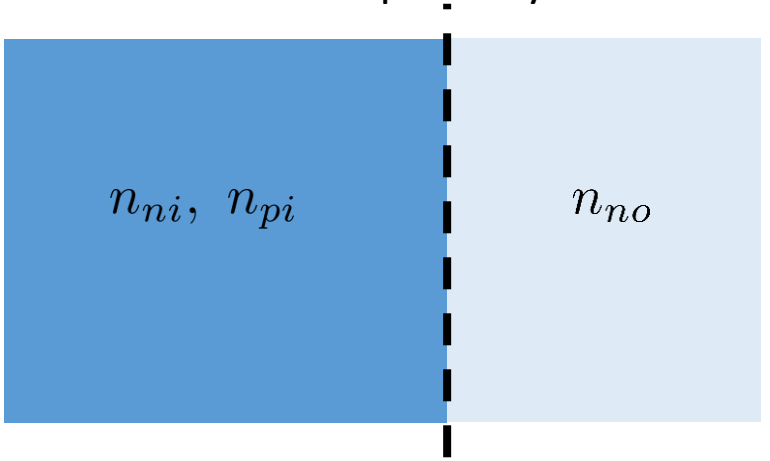
Basic element of CLDM is a prescription to calculate the surface properties (=correction to the reference energy).
It should be:

- Tractable
- Enough accurate

# Surface energy is natural. It can be calculated

Realistic two-phase system

Reference two-phase system




$$E_{\text{two phase}}, N_{\text{two phase}}, Z_{\text{two phase}}$$

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Can be nulled by choose of the reference system

<u>Crucial element:</u>

- parametrization of the surface energy

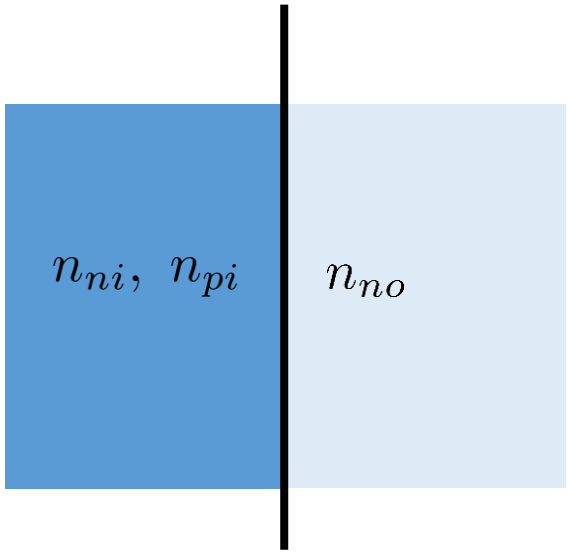
Most of previous works: $x_p = n_{pi}/n_{ni}$

Here: $\mu_n \leftrightarrow \nu_s$

# Equilibrium of the plain interface

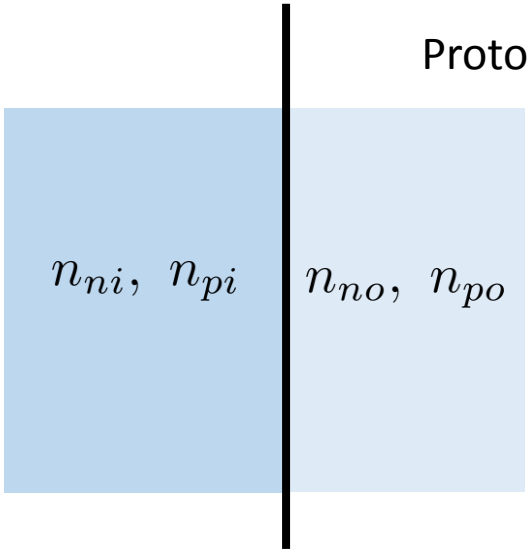

Vacuum

$$\mu_n < m_n$$
$$\mu_p < m_p$$
$$P = 0$$



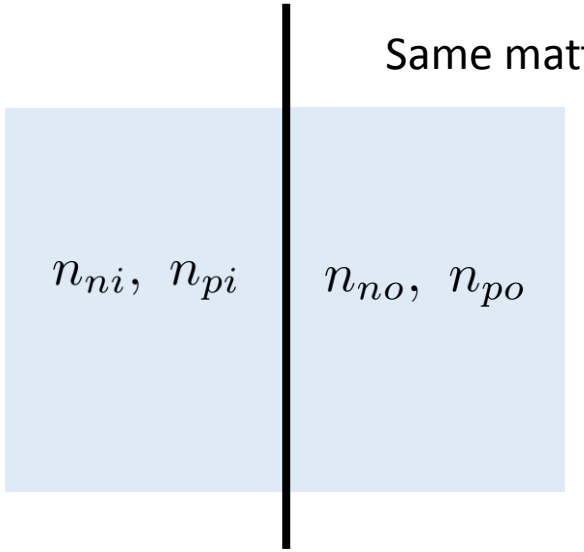

Neutronization

$$\mu_{pi} < \mu_{po}$$
$$\mu_{ni} = \mu_{no}$$
$$P_i = P_o$$




Proton drip

$$\mu_{pi} = \mu_{po}$$
$$\mu_{ni} = \mu_{no}$$
$$P_i = P_o$$




Same matter on both sides

$$n_{pi} = n_{po}$$
$$n_{ni} = n_{no}$$
$$\nu_s = 0$$
$$\sigma = 0$$

Keller et al., arXiv:2401.13461: unavoidable

Two phase equilibrium is governed by one parameter (for plain interface)

# How to construct CLDM for inner crust?

Thermodynamics

Consider a cell thermodynamically:

- Write a general expression for cell energy
- Present the surface terms as

$$E_{\rm surf} = E_{\rm two\ phase} - E_{\rm ref}$$
$$N_{\rm surf} = N_{\rm two\ phase} - N_{\rm ref}$$
$$Z_{\rm surf} = Z_{\rm two\ phase} - Z_{\rm ref}$$

- Consider thermodynamic derivatives
- Obtain relations for surface terms

Perturbative ETF

- Apply plane-boundary nucleon distribution profiles as a zero-approximation
- Consider small corrections perturbatively, threating surface curvature as perturbation parameter.
- ***Up to the second order in surface curvature all parameters of the theory can be determined from the plane-boundary nucleon profiles***

(it is due to optimal choice of plane boundary nucleon profiles: small perturbations of these profile should not affect energy for plane boundary)

# Surface energy

$$E_s = \sigma S + \mu_{\mathrm{ns}} N_{\mathrm{s}}$$

Thermodynamic relation: $$\nu_s \equiv \frac{N_s}{S} = -\frac{\partial \sigma}{\partial \mu_{\mathrm{ns}}}$$

$$\varepsilon_{\mathrm{s}} \equiv \frac{E_S}{S} = \left(\sigma_{\mathrm{pl}}\left(\mu_{\mathrm{ns}}\right) + \frac{2\sigma_{\mathrm{cur}}\left(\mu_{\mathrm{ns}}\right)}{r_{\mathrm{p}}}\right)\frac{3u}{r_{\mathrm{p}}} + \left(\nu_{\mathrm{pl}}\left(\mu_{\mathrm{ns}}\right) + \frac{2\nu_{\mathrm{cur}}\left(\mu_{\mathrm{ns}}\right)}{r_{\mathrm{p}}}\right)\frac{3u\mu_{\mathrm{ns}}}{r_{\mathrm{p}}}$$

Surface parameters:

$$\sigma_{pl} = \int_{-\infty}^{\infty} \left\{\left[\epsilon(z) - \epsilon_{ref}(z)\right] - \mu_n\left[n_n(z) - n_{n,ref}(z)\right]\right\} \mathrm{d}z$$

$$\sigma_{cur} = \int_{-\infty}^{\infty} \left\{\left[\epsilon(z) - \epsilon_{ref}(z)\right] - \mu_n\left[n_n(z) - n_{n,ref}(z)\right]\right\} z\mathrm{d}z$$

# Structure of compressible liquid drop model

$$\epsilon = u\,\epsilon^{\rm nm}(n_{ni},\, n_{pi}) + (1-u)\,\epsilon^{\rm nm}(n_{no},\, 0) + \frac{E_s(\nu_s, r_p)}{V_c} + \frac{E_C(n_{pi}, r_p, w)}{V_c} + \epsilon_e(n_e).$$

Explicit (algebraic) expression for the energy density

6 parameters:

$$n_{ni},\, n_{pi},\, n_{no},\, \nu_s,\, V_c,\, r_p$$

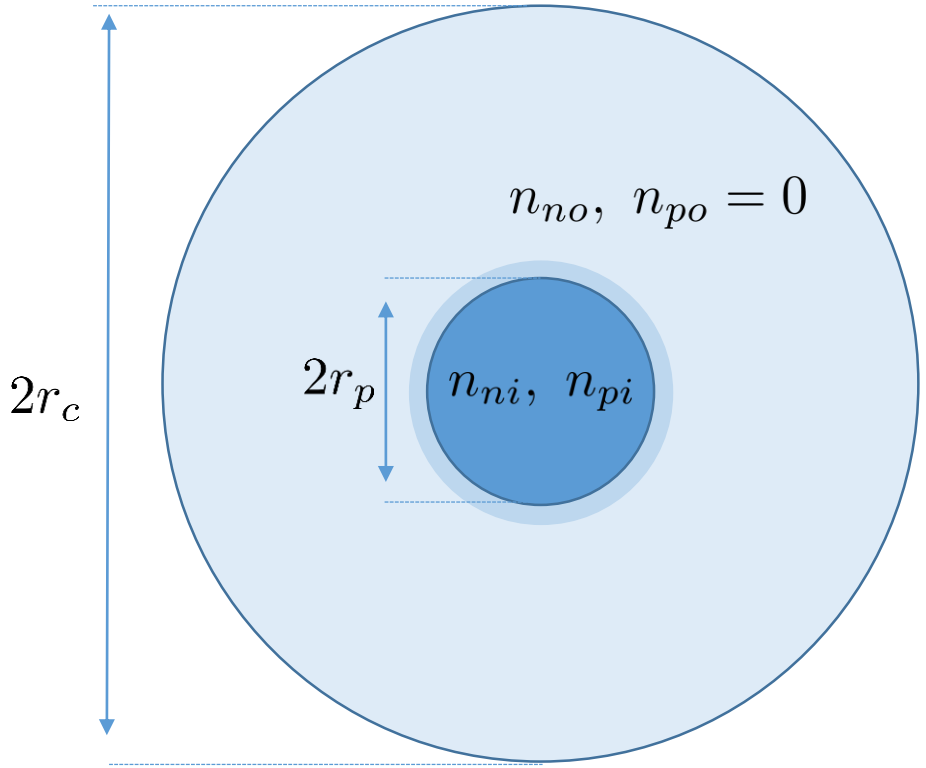
Minimization at fixed $n_b$

System of 5 algebraic (nonlinear) equations
With clear physical meaning:

- Chemical equilibrium inside the cell (2 eqs.)
- Beta-equilibrium (1 eq)
- Mechanical equilibrium (1 eq)
- Optimal size of the cell (1 eq)

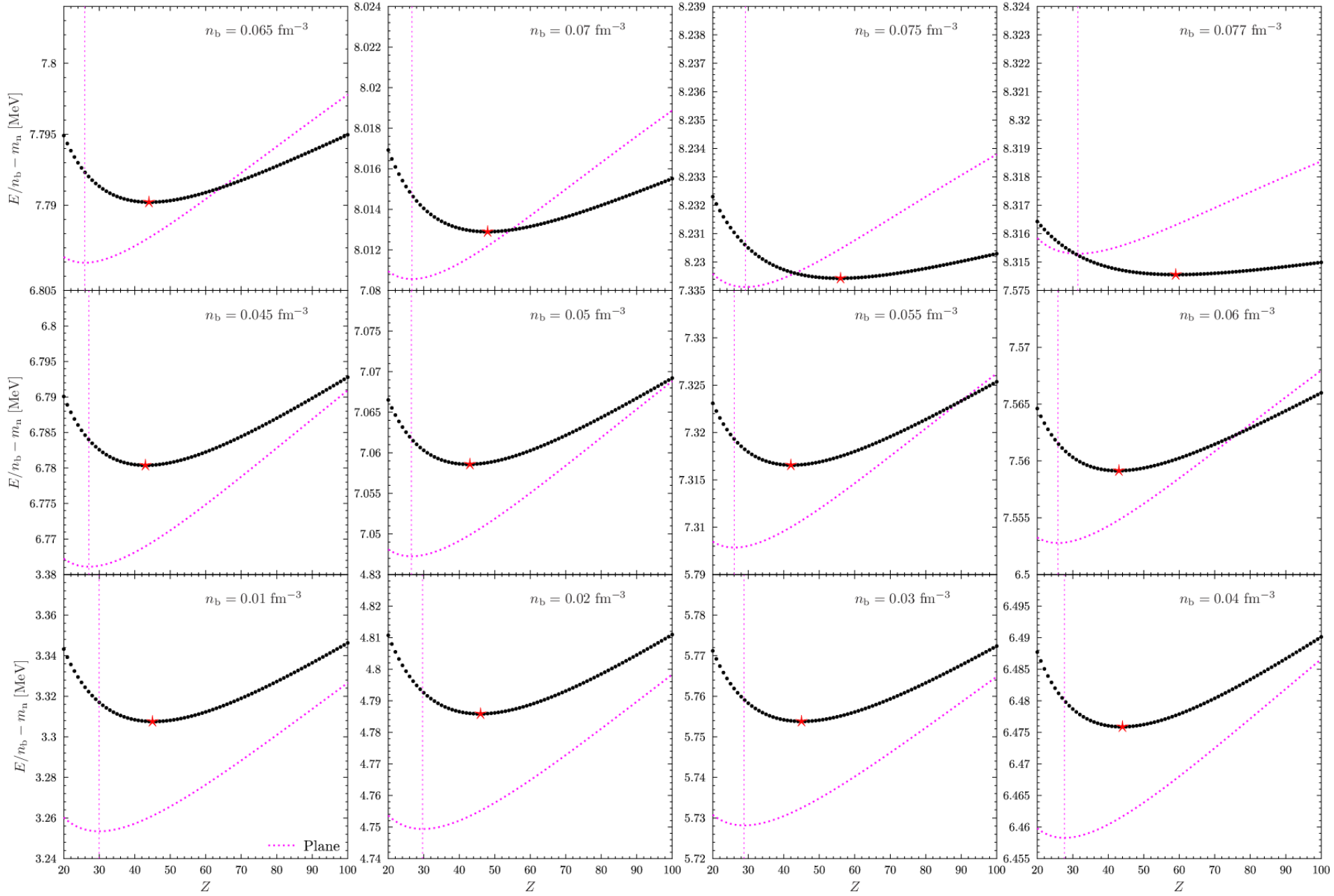
Explicit formulae for thermodynamic quantities – absolute thermodynamic consistency!

$$\mu_n,\; P,\; \ldots$$

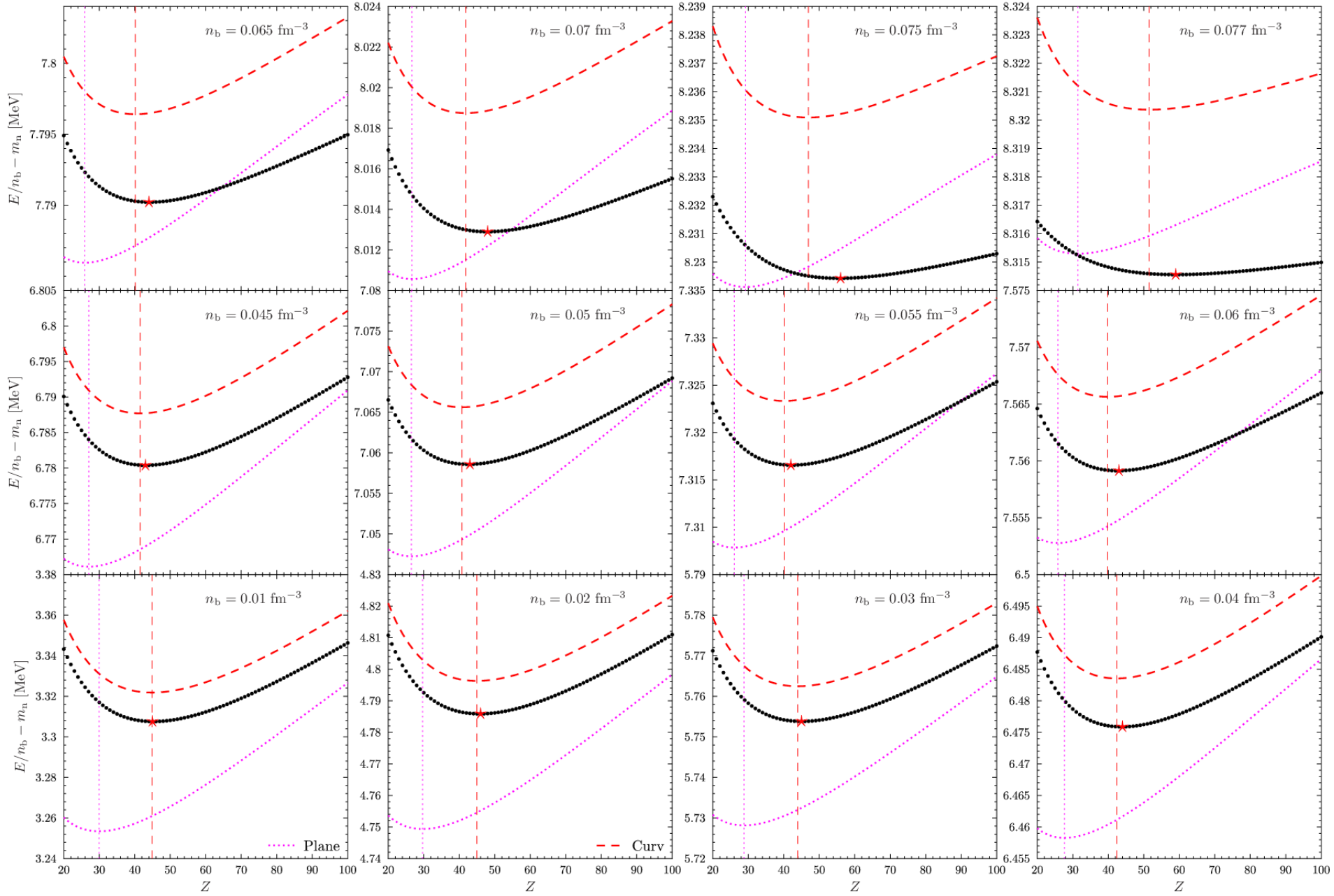



Without curvature corrections: Gusakov, AIC (2020)

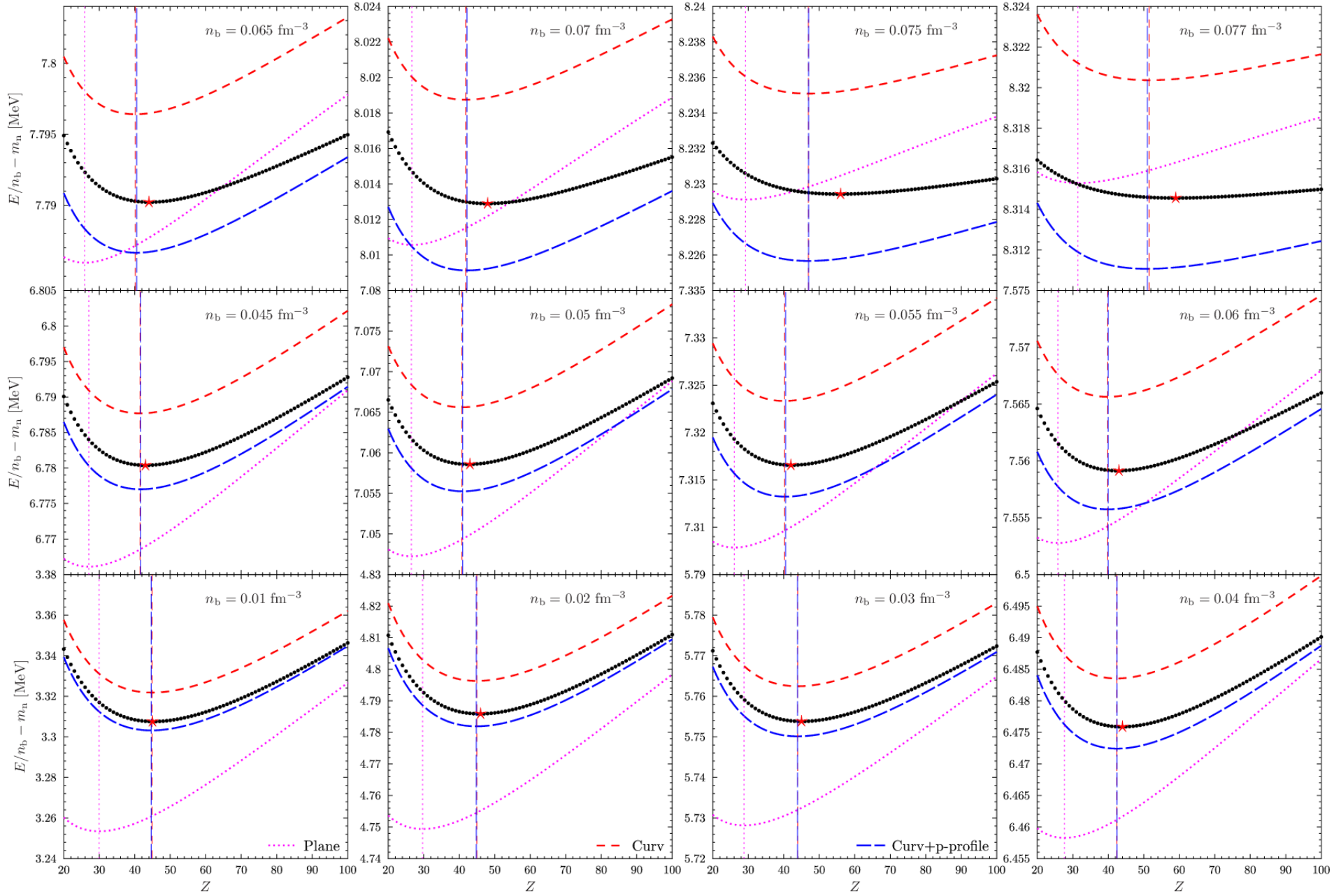
# CLDM vs ETF: no curvature corrections (Gusakov&AIC 2020)



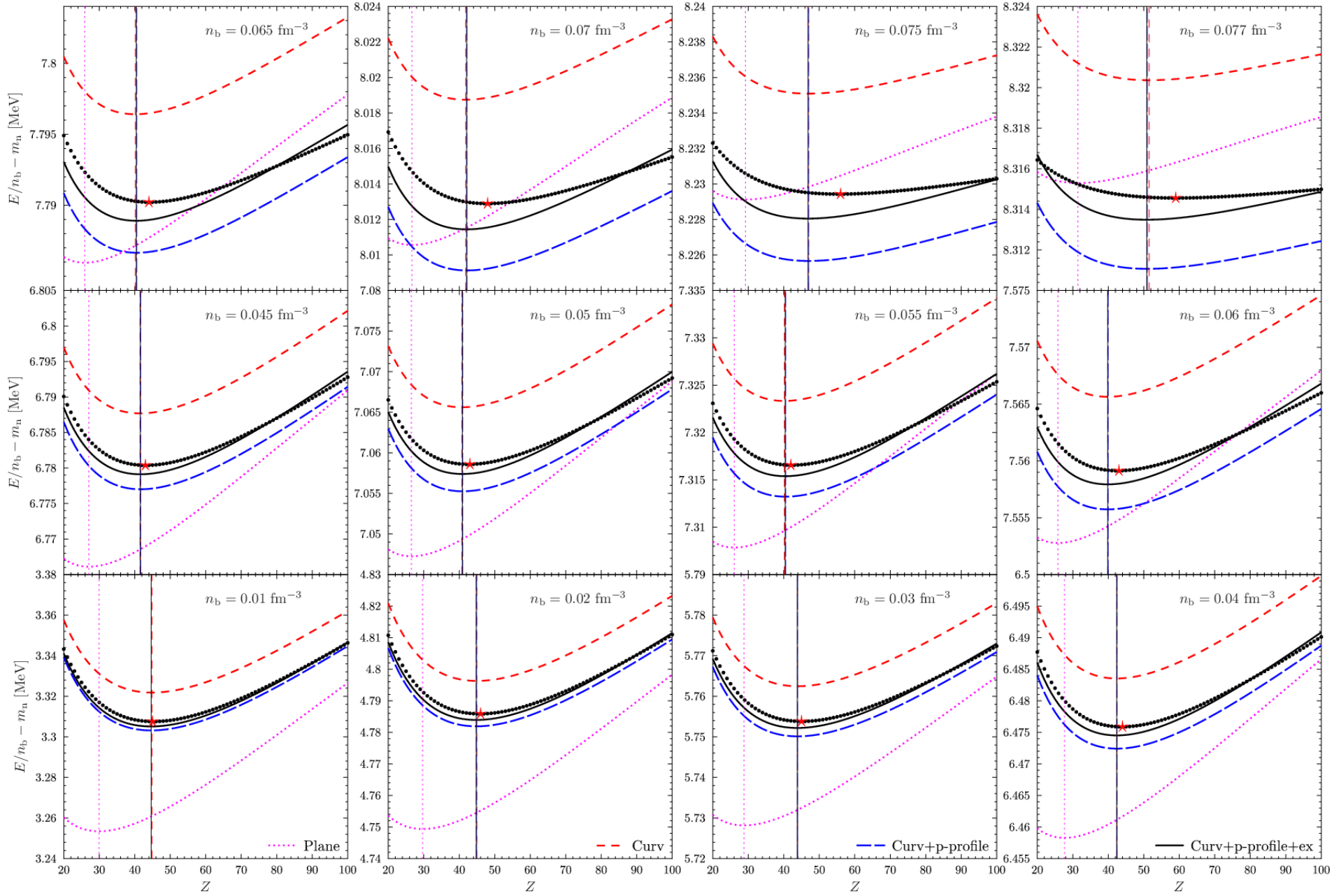
# CLDM vs ETF: curvature corrections



# CLDM vs ETF: curvature + p-profile (Coulomb energy)



# CLDM vs ETF: curvature + p-profile+ exchange cor.



# Conclusions

**We suggested compressible liquid drop model (CLDM) for neutron star inner crust, which reproduce ETF results with accuracy ~2 keV/nucleon**

Note: the accuracy ~2 keV/nucleon is theoretical, i.e. it corresponds to comparison of ETF and CLDM calculations for the same nucleon interaction potential (BSK24 in the figures)

- CLDM can be incorporated into the codes
- CLDM was constructed on the base of ETF approach.
- It was done in two ways: thermodynamically and perturbatively.
- Being constructed pertubatively, CLDM can be applied to infer nucleon distribution profiles. This feature, potentially, can be used to add shell corrections via Stutinsky integral theorem (to be done…)

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