The effect of spectroscopic factors on the alpha decay of actinide nuclei.

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α-decay as a tunneling process

- α-particle is formed on the surface of daughter nucleus with probability given by the spectroscopic factor $S(\alpha)$.
- After formation, α-particle tunnels through the barrier in the interaction potential.

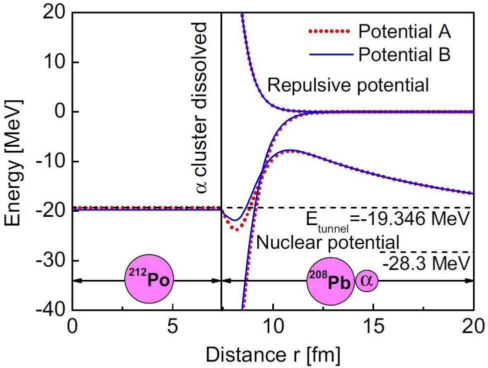
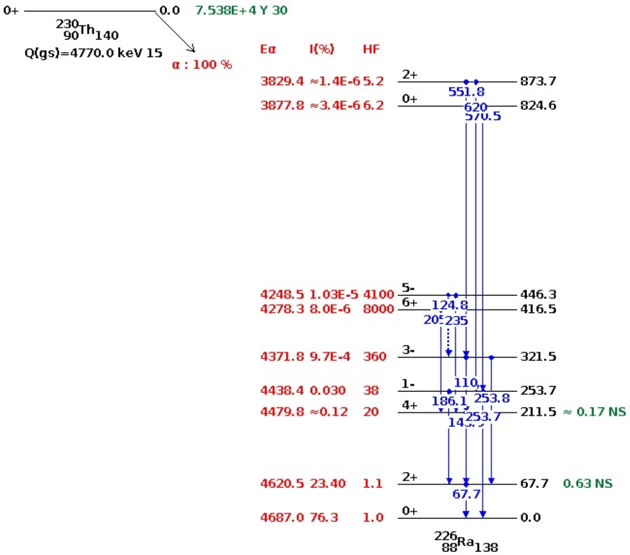


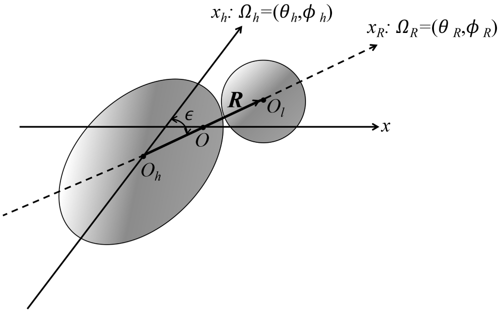
Figure from: P. Schuck, et al., Phys. Scr. **91**, *123001 (2016).*

Fine structure in α-decay



Degrees of Freedom of DNS

In α-decay, nucleus can be treated as a DNS with α-particle and daughter nucleus in touching.



- rotation of the system as a whole $\Omega_R = (\theta_R, \phi_R)$
- rotation of the deformed fragment $\Omega_h = (\theta_h, \phi_h)$
- relative motion in R
- motion in mass (charge)-asymmetry

Daughter nucleus can be deformed. It is assumed it has axially-symmetric quadrupole β_2 and octupole β_3 deformations.

Interaction potential of α+daughter nucleus

$$V(R,\varepsilon) = V_N(R,\varepsilon) + V_C(R,\varepsilon)$$

$$V_C(R,\varepsilon) = \sum_\lambda (-1)^\lambda \frac{e^2 Z_1 Z_2}{R^{\lambda+1}} Q_\lambda^{(1)} P_\lambda(\cos\varepsilon), \quad Q_\lambda^{(i)} = \frac{1}{Z_i}\int \rho_i(\mathbf{r}) r^\lambda P_\lambda(\cos\varepsilon) d\mathbf{r}$$

Double-folding potential: (G. G. Adamian *et al.*, IJMPE **5**, 191 (1996).)

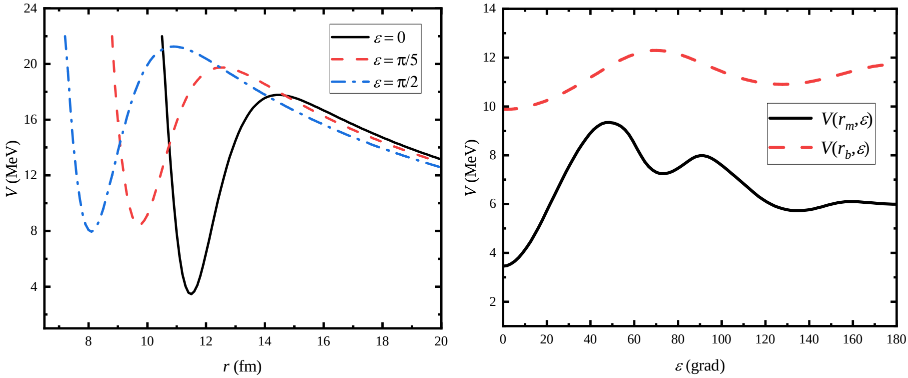
$$V_N(R,\varepsilon) = \int \rho_1(\mathbf{r}_1)\rho_2(\mathbf{r}_2) F((R,\varepsilon) + \mathbf{r}_2 - \mathbf{r}_1)\mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2,$$

with Migdal forces: A. B. Migdal, *Theory of finite Fermi Systems...* (Nauka, 1982).

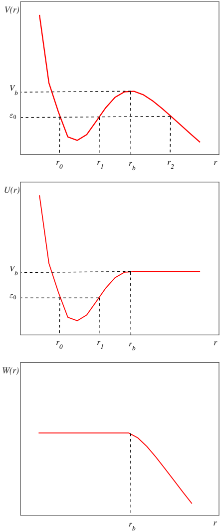
$$F(\mathbf{R}+\mathbf{r}_2-\mathbf{r}_1) = C_0\left(F_{in}\frac{\rho_1(\mathbf{r}_1)+\rho_2(\mathbf{r}_2)}{\rho_{00}} + F_{ex}\left(1-\frac{\rho_1(\mathbf{r}_1)+\rho_2(\mathbf{r}_2)}{\rho_{00}}\right)\right)\delta(\mathbf{R}+\mathbf{r}_2-\mathbf{r}_1),$$

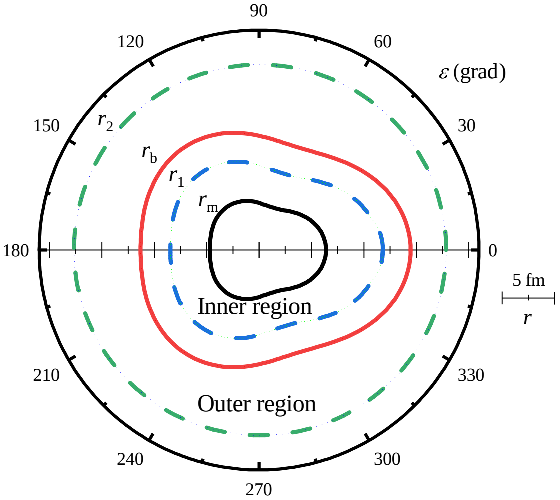
$F_{in,ex} = (f_{in,ex} + f'_{in,ex}\tau_\mathbf{1}\cdot\tau_\mathbf{2}) + (g_{in,ex} + g'_{in,ex}\tau_\mathbf{1}\cdot\tau_\mathbf{2})\sigma_\mathbf{1}\cdot\sigma_\mathbf{2}$
$C_0 = 300\ \mathrm{MeV\ fm}^3,\ f_{in,ex} = 0.09(-2.59),\ f'_{in,ex} = 0.42(0.54)$

Interaction potential of α+daughter nucleus



Inner and outer regions for alpha-decay process





Two-potential approach to tunneling problem

$$H = H_0 + W(r), \quad H_0 = -\frac{\hbar^2 \nabla^2}{2\mu} + U(r)$$

$\Phi_0(r)$ is the bound state of Hamiltonian H_0

$\Phi_k(r)$ are continuum states of Hamiltonian H_0

$$H_0 \Phi_0(r) = \varepsilon_0 \Phi_0(r), \quad H_k \Phi_k(r) = (V_b + \varepsilon_k) \Phi_k(r)$$

Assuming that perturbation $W(r)$ switched on at $t = 0$

$$\Psi_0(r,t) = b_0(t) e^{-\frac{i\varepsilon_0 t}{\hbar}} \Phi_0(r) + \int \frac{d^3k}{(2\pi)^3} b_k(t) e^{-\frac{i(V_b+\varepsilon_k)t}{\hbar}} \Phi_k(r)$$

S. A. Gurvitz, PRA **38**, 1748 (1988); **69**, 042705 (2004).

Two-potential approach to tunneling problem

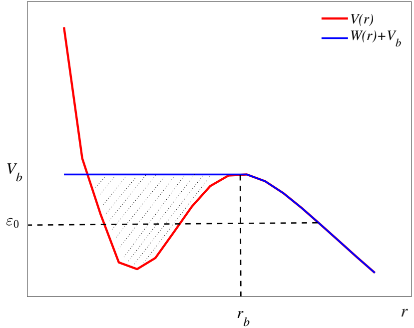
$$\varepsilon = \langle \Phi_0 | W | \Phi_0 \rangle + \langle \Phi_0 | W G_{\tilde{W}}(\varepsilon_0) W | \Phi_0 \rangle = \varepsilon_0 + \Delta - i\Gamma/2$$

Decay width

$$\Gamma = -2\mathrm{Im}(\varepsilon_0)$$

Green's function:

$$G_{\tilde{W}}(\varepsilon) = \left[\varepsilon + \frac{\hbar^2 \nabla^2}{2\mu} - W - V_b \right]^{-1}$$



$$\Gamma = \frac{4\mu}{\hbar^2 k} \left| \int_{r_b}^{\infty} \Phi_0(r) W(r) \chi_k(r) dr \right|^2$$

Hamiltonian for the α-particle DNS

$$\hat{H} = \hat{T} + U(r, \Omega_h, \Omega_R),$$

Kinetic energy operator:

$$\hat{T} = -\frac{\hbar^2}{2\mu r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\hbar^2}{2\mu r^2} l_R^2 + \frac{\hbar^2}{2\Im_h} l_h^2,$$

Angular momentum operators:

$$l_i^2 = -\frac{1}{\sin\theta_i}\frac{\partial}{\partial\theta_i}\sin\theta_i\frac{\partial}{\partial\theta_i} - \frac{1}{\sin^2\theta_i}\frac{\partial^2}{\partial\phi_i^2}$$

Potential energy is calculated first in the body-fixed system:

$$U(r, \varepsilon) = U_0(r) + U_2(r)P_2(\cos\varepsilon) + U_3(r)P_3(\cos\varepsilon),$$

and then transformed into laboratory system:

$$P_\lambda(\cos\varepsilon) = \frac{(-1)^\lambda}{\sqrt{2\lambda+1}}\left[Y_\lambda(\Omega_H)\times Y_\lambda(\Omega_R)\right]_{(00)} \equiv \frac{(-1)^l}{\sqrt{2\lambda+1}}\left[\lambda\times\lambda\right]_{(00)}.$$

Wave function of the α-particle DNS: radial motion

Taking into account that the motion in r is much faster than the motion in angular variables:

$$\hbar\omega_R \gg \hbar^2/\Im_H, \hbar^2/(\mu r_m^2).$$

the wave-function of can be searched for in B.-O. approximation:

$$\Phi_\alpha(r, \Omega_h, \Omega_R) = \frac{\psi(r)}{r}\mathcal{Y}(\Omega_h, \Omega_R).$$

Equation for the radial wave-function:

$$\left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2} + U(r, \Omega_h, \Omega_R)\right]\psi(r) = E(\Omega_h, \Omega_R)\psi(r).$$

Energy $E(\Omega_h, \Omega_R)$ plays a role of effective-potential for angular motion. It can be expanded in bipolar spherical functions:

$$E(\Omega_h, \Omega_R) = \langle T_r \rangle + \langle U(r_m(\varepsilon), \varepsilon) \rangle = C_0 + C_2\,[2 \times 2]_{(00)} + C_3\,[3 \times 3]_{(00)}\,,$$

Wave function of the α-particle DNS: angular motion

For angular wave function we have following equation:

$$\left(\frac{\hbar^2 \hat{I}_R^2}{2\mu r_m^2} + \frac{\hbar^2 \hat{I}_H^2}{2\Im_H} + C_0 + C_2\,[2\times 2]_{(00)} + C_3\,[3\times 3]_{(00)}\right)\mathcal{Y}(\Omega_{h,R}) = Q_\alpha \mathcal{Y}(\Omega_{h,R}).$$

It can be diagonalized on a set of bipolar spherical functions:

$$\mathcal{Y}(\varepsilon) = \sum_\lambda a_\lambda\,[Y_\lambda(\Omega_H)\times Y_\lambda(\Omega_R)]_{(00)} = \sum_\lambda a_\lambda (-1)^\lambda \sqrt{\frac{2\lambda+1}{2}} P_\lambda(\cos\varepsilon).$$

Squares of amplitudes a_λ gives a probability to find an alpha-particle DNS in the state where the daughter nucleus are excited into the state with angular momentum λ. The dependence of the spectroscopic factor on angular momentum can be defined as

$$S_\lambda = S|a_\lambda|^2.$$

Perturbation theory for deformed potential

Shrödinger equation:

$$\left(H_0 + \frac{\hbar^2 l(l+1)}{2\mu r^2} + V_2(r)\langle l|2|l\rangle - Q\right)\psi_l(r) = -\sum_{l'} V_2(r)\langle l'|2|l\rangle\psi_{l'}(r).$$

As basis:

$$\left(H_0 + \frac{\hbar^2 l(l+1)}{2\mu r^2} + V_2(r)\langle l|2|l\rangle - Q\right)\psi_l^{(0)}(r) = 0$$

First-order correction, using Green function:

$$\psi_l^{(1)}(r) = -\frac{2\mu}{\hbar^2}\int W(r')G_{kl}(r,r')dr'$$

Green function:

$$G_{kl}^{(\pm)}(r,r') = -\frac{1}{k}\chi_{kl}^{(\pm)}(r_>)\chi_{kl}(r_<)$$

Width of α-decay

$$\Gamma = \frac{\hbar^2}{4\mu k} \sum_l \Gamma_l$$

- Zero-order term:

$$\Gamma_l^{(0)} = S_l^2 j_{kl}^2 + S_l^2 \frac{4\mu}{\hbar^2 k} (l0|20|l0)^2 \langle kl|W_2|kl\rangle j_{kl}^2$$

- Interference term:

$$\Gamma_l^{(1)} = 2 \sum_{l' \neq l} S_l S_{l'} \frac{2\mu}{\hbar^2 k} \sqrt{\frac{2l'+1}{2l+1}} (l'0|\lambda 0|l0)^2 \langle kl|W|k'l'\rangle j_{kl} j_{k'l'}$$

$$j_{kl} = (\psi'(r)\chi_{kl}(r) - \psi(r)\chi'_{kl}(r))|_{r=r_b(\varepsilon)}$$

Width of α-decay

In quasiclassical approximation:

$$\chi_{k_0 l}(r) = \frac{\sqrt{k_0}}{2\sqrt{|p_l(r)|}} \exp\left(-\int_r^{r_2(l,\varepsilon)} |p_l(r')| dr'\right)$$

, where

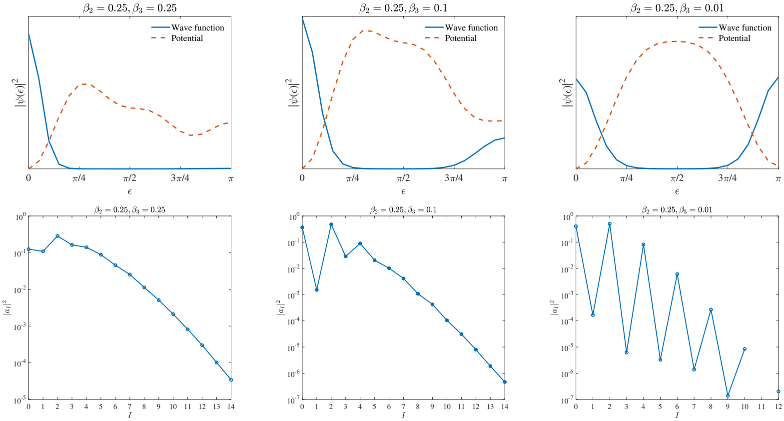
$$p_l(r) = \sqrt{\frac{2\mu}{\hbar^2}\left(Q_0 - E_d(l) - V_{ll}(R) - \frac{\hbar^2 l(l+1)}{2\mu r^2}\right)}$$

$$V_{ll}(r) = \frac{2l+1}{2}\int_0^{\pi} P_l^2(\cos\varepsilon) V(r,\varepsilon)\sin(\varepsilon) d\varepsilon$$

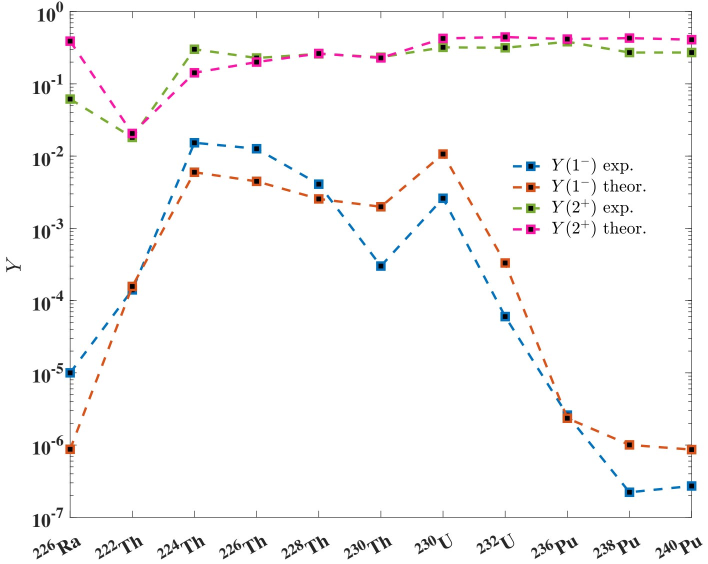
As a result:

$$\Gamma_l^{(0)} = S_l^2 \frac{\hbar\omega_0}{2\pi} \exp\left[-\frac{2}{\hbar}\int_{r_1}^{r_2} \sqrt{2\mu(Q - V(r'))} dr'\right]$$

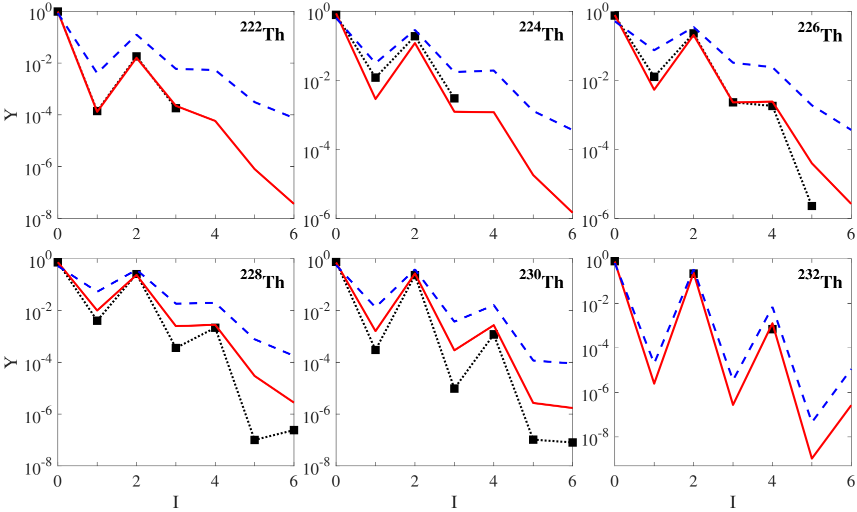
Structure of wave functions of α-particle DNS



Decay to the lowest excited states



Fine structure for α-decay in Th isotopes



Conclusion

- We propose a model which allows to describe the alpha-decay fine structure of actinides and transactinides nuclei, including an additional hindrance on the decay to the states with odd angular momentum.
- To describe the tunneling of alpha-particle, the two-potential approach was used.
- The bound state of nucleus before decay, was described as the dinuclear system consists of daughter nucleus and alpha-particle. The wave function of α-DNS was calculated as a superposition of different angular momentum components.

Thank You!

Two-potential approach to tunneling problem

$$i\hbar\frac{db_0(t)}{dt} = b_0(t)\langle\Phi_0|W|\Phi_0\rangle + \int \frac{d^3k}{(2\pi)^3} b_k(t) e^{-\frac{i(\varepsilon_0 - V_b - \varepsilon_k)t}{\hbar}} \langle\Phi_0|W|\Phi_k\rangle$$

$$\begin{aligned} i\hbar\frac{db_k(t)}{dt} &= b_0(t)\langle\Phi_k|W|\Phi_0\rangle e^{-\frac{i(V_b-\varepsilon_0-\varepsilon_k)t}{\hbar}} \\ &+ \int \frac{d^3k'}{(2\pi)^3} b_{k'}(t) e^{-\frac{i(\varepsilon_k - \varepsilon_{k'})t}{\hbar}} \langle\Phi_k|W|\Phi_{k'}\rangle \end{aligned}$$

Neglecting matrix elements $\langle\Phi_k|W|\Phi_{k'}\rangle$, one can obtain Fermi golden rule for decay width:

$$\Gamma = 2\pi \int |\langle\Phi_k|W|\Phi_0\rangle|^2 \rho(\varepsilon_k + V_b)\delta(\varepsilon_0 - \varepsilon_k) d\varepsilon_k$$

Two-potential approach to tunneling problem

Applying Laplace transformation

$$b(\varepsilon) = \int_0^\infty e^{\frac{i\varepsilon t}{\hbar}} b(t)dt, \quad b_0(t) = \frac{1}{2\pi\hbar}\int_C d\varepsilon e^{\frac{-i\varepsilon t}{\hbar}} b_0(\varepsilon)$$

$$b_0(\varepsilon) = \frac{i\hbar}{\varepsilon - \langle\Phi_0|W|\Phi_0\rangle - \langle\Phi_0|W\tilde{G}W|\Phi_0\rangle}$$

where Green function $\tilde{G}$ satisfies

$$\tilde{G} = \tilde{G}_0 + \tilde{G}_0(W + V_b)\tilde{G},$$

$$\tilde{G}_0 = \sum_{\mathbf{k}} \frac{|\Phi_{\mathbf{k}}\rangle\langle\Phi_{\mathbf{k}}|}{\varepsilon + \varepsilon_0 - \varepsilon_k}$$

$$\Gamma = -2\mathrm{Im}(\varepsilon)$$

Wave function in deformed potential

$$\psi_{kl} = \psi_{kl}^{0} + \sum_{l'} \frac{2m}{\hbar^2 k} \sqrt{\frac{2l'+1}{2l+1}} (l'0|\lambda 0|l0)^2 \langle kl|W_{rad}|k'l'\rangle \psi_{k'l'}^{0}$$

$$\psi_{kl}^{0} = \psi_{kl}^{coul} + \frac{2m}{\hbar^2 k} (l0|20|l0)^2 \langle kl|W_2|kl\rangle \psi_{kl}^{coul}$$

$$\psi_{kl} = \psi_{kl}^{coul} + \frac{2m}{\hbar^2 k} (l0|20|l0)^2 \langle kl|W_2|kl\rangle \psi_{kl}^{coul} +$$

$$+ \sum_{l'} \frac{2m}{\hbar^2 k} \sqrt{\frac{2l'+1}{2l+1}} (l'0|\lambda 0|l0)^2 \langle kl|W_{rad}|k'l'\rangle \psi_{k'l'}^{coul} +$$

$$+ \frac{2m}{\hbar^2 k} (l0|20|l0)^2 \sum_{l'} \frac{2m}{\hbar^2 k} \sqrt{\frac{2l'+1}{2l+1}} (l'0|\lambda 0|l0)^2 \langle kl|W_{rad}|k'l'\rangle \psi_{kl}^{coul} \langle kl|W_2|$$